

# Dupin Cyclide Blends Between Quadric Surfaces for Shape Modeling

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## Abstract

We introduce a novel method to define Dupin cyclide blends between quadric primitives. Dupin cyclides are non-spherical algebraic surfaces discovered by French mathematician Pierre-Charles Dupin at the beginning of the 19th century. As a Dupin cyclide can be fully characterized by its principal circles, we have focussed our study on how to determine principal circles tangent to both quadrics being blended. This ensures that the Dupin cyclide we are constructing constitutes a  $G^1$  blend. We use the Rational Quadratic Bézier Curve (RQBC) representation of circular arcs to model the principal circles, so the construction of each circle is reduced to the determination of the three control points of the RQBC representing the circle.

In this work, we regard the blending of two quadric primitives  $A$  and  $B$  as two complementary blending operations: primitive  $A$ -cylinder and cylinder-primitive  $B$ ; two Dupin cyclides and a cylinder are then defined for each blending operation. In general the cylinder is not useful and may be reduced to a simple circle. A complete shape design example is presented to illustrate the modeling of Eurographics'04 Hugo using a limited number of quadrics combined using Dupin cyclide blends.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

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## 1. Introduction

Dupin cyclides are non-spherical algebraic surfaces discovered by French mathematician Pierre-Charles Dupin at the beginning of the 19th century [Dup22]. These surfaces have an essential property: all their curvature lines are circular.

R. Martin, in his 1982 PhD thesis, is the first author who thought of using these surfaces in CAD/CAM and geometric modeling [Mar82], using cyclides in the formulation of his principal patches. Afterwards, Dupin cyclides gained a lot of attention, and their algebraic and geometric properties have been studied in depth by many authors [BG92, Ber78, Heb81, Pin85, Ban70, DMP93, Pra97] which has allowed to see the adequacy and the contribution of these surfaces to geometric modeling.

From the CAD/CAM point of view, a cyclide is an algebraic surface of low degree having an easy to understand parametric representation (with 3 parameters) as well as two equivalent implicit equations [Pra97, For12].

Their low degree algebraic equation and simple parametric representation give cyclides advantages of both implicit and parametric formulations. The low algebraic degree allows the development of robust and quick geometric computation algorithms (point on surface, derivatives, tangents, lines of curvatures, etc.). The parametric form allows easy editing and quick visualization of these surfaces. Another important point that marks the beauty of Dupin cyclides is their well-established conversions to other parametric surfaces such as Bézier, Bspline, or NURBS [Pra97, Boe90, AD96, Deg94, ZS92, SD95]. This conversion of Dupin cyclides into rational quadric Bézier patches bridges the gap between these two modeling tools

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and may accelerate the introduction of cyclides into geometric modeling systems based on classical parametric surfaces.

This paper introduces a novel method to define Dupin cyclide blends between quadric surfaces. As it is well established that the construction of a Dupin cyclide is totally dependant on its principal circles, we have focussed our attention on how to determine principal circles tangent to both quadrics being blended. This ensures that the Dupin cyclide we are constructing constitutes a  $G^1$  blend. We use the Rational Quadratic Bézier Curves (RQBCs) representation of circular arcs to model the principal circles, so the construction of each and every circle is reduced to the determination of the three control points of the RQBC representing the circle. For short, in the rest of this paper we refer to Dupin cyclides as cyclides.

In this work, we regard the blending of two quadric primitives  $A$  and  $B$  as two complementary blending operations primitive  $A$ -cylinder and cylinder-primitive  $B$ , two Dupin cyclides and a cylinder are then defined for each blending operation. In general the cylinder is not useful and may be reduced to a simple circle. By proceeding this way we aim at increasing the flexibility of quadric combination for shape design and limiting the number of particular blends; for example, a complete combination of four quadric primitives (cylinder, cone, sphere and hyperboloid) requires the definition of ten variants of the blending algorithm, while employing our approach requires only four variants (cylinder-cylinder, cylinder-cone, cylinder-sphere and cylinder-hyperboloid).

The paper is organized as follows. Section 2 reviews some necessary material needed in the rest of the paper. Definitions of cyclides, Dupin cyclides, RQBCs together with the representation of circular arcs using RQBCs. Sections 3, 4, 5 and 6 describe our quadric-quadric blending algorithm. For lack of place, only a subset of blending cases is considered: cylinder-cyclide (Sec. 3), cylinder-cone (Sec. 4), cylinder-sphere (Sec. 5) and cone-sphere (Sec. 6). A complete shape design example is presented in section 7: it illustrates the modeling of the Eurographics'04 Hugo using a limited number of quadrics combined with Dupin cyclide blends. The last section concludes with a discussion of this work's contribution and of suggested perspectives for future research.

## 2. Background

### 2.1. The cyclide surfaces

Anallagmatic surfaces are surfaces invariant by at least one inversion. They are the envelope surfaces of a moving sphere, which is orthogonal to a given fixed sphere (called the directrix sphere), and the centres of which lie on a given quadric surface [Dar73, Dar17]. Moutard proved in 1804 that cyclides are a special kind of anallagmatic surfaces: they are invariant by five inversions [Mou04]. Two equivalent im-

PLICIT equations characterize the cyclides of degree 4:

$$r(x, y, z)^4 + 2P_1(x, y, z) \times r(x, y, z)^2 + P_2(x, y, z) = 0 \quad (1)$$

$$\left( r(x, y, z)^2 + P_1(x, y, z) \right)^2 = P_2'(x, y, z) \quad (2)$$

where  $P_1$  is a homogeneous polynomial of degree 1,  $P_2$  and  $P_2'$  are polynomials of degree 2 and  $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ . Cyclides of degree 3 are the images of cyclides of degree 4 by an inversion. Dupin cyclides are a special kind of cyclides: their spine curve is a conic curve, instead of a quadric surface [Dar73, Dar17].

### 2.2. Dupin cyclides

It is possible to define Dupin cyclides in various equivalent ways: Liouville showed that a Dupin cyclide is the image of a circular cone, a circular cylinder or a torus under an inversion; A Dupin cyclide is the envelope of spheres centered on a given conic and orthogonal to a given fixed sphere, called sphere of inversion: the latter is centered on the focal axis of the cyclide [Dar73, Dar17]; A Dupin cyclide is the envelope surface of a set of spheres, the centers  $M$  of which lies on a given conic with focus  $F$ , and the radius  $R$  of which is such that the distance  $FM + R$  is a given constant (this definition is due to Maxwell); A Dupin cyclide is the envelope surface of the spheres tangent to three given fixed spheres [Dup22].

Dupin cyclides of degree 4 depend on three independent parameters  $a$ ,  $c$  and  $\mu$  with  $a^2 > c^2$ . It is convenient to define  $b = \sqrt{a^2 - c^2}$ . Dupin cyclides have a parametric equation:

$$\begin{cases} x(\theta; \psi) = \frac{\mu(c - a \cos \theta \cos \psi) + b^2 \cos \theta}{a - c \cos \theta \cos \psi} \\ y(\theta; \psi) = \frac{b \sin \theta \times (a - \mu \cos \psi)}{a - c \cos \theta \cos \psi} \\ z(\theta; \psi) = \frac{b \sin \psi \times (c \cos \theta - \mu)}{a - c \cos \theta \cos \psi} \end{cases} \quad (3)$$

where  $(\theta; \psi) \in [0; 2\pi]^2$ , and two implicit equations:

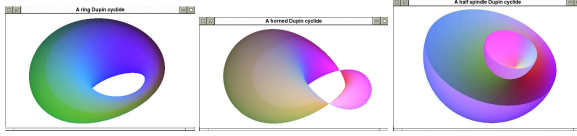
$$(x^2 + y^2 + z^2 - \mu^2 + b^2)^2 - 4(ax - c\mu)^2 - 4b^2y^2 = 0 \quad (4)$$

$$(x^2 + y^2 + z^2 - \mu^2 - b^2)^2 - 4(cx - a\mu)^2 + 4b^2z^2 = 0 \quad (5)$$

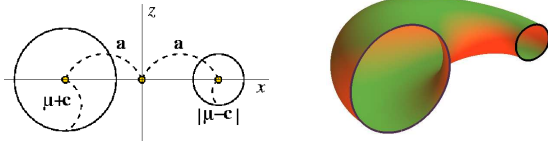
Implicit equations were obtained by Darboux [Dar17] early in the last century by computing the implicit equation of the envelope of the spheres defining the Dupin cyclide. The same results are also given by Forsyth [For12].

According to the parameter values, there are three kinds of Dupin cyclides: ring cyclide, horned cyclide and spindle cyclide. Figure 1 shows examples of half Dupin cyclides.

A Dupin cyclide admits two symmetry planes  $P_y : (y = 0)$  and  $P_z : (z = 0)$ . The section of a Dupin cyclide by one of these two planes is the union of two circles called principal circles. Two of these principal circles are sufficient to determine the three parameters  $a$ ,  $c$  and  $\mu$ , and thus to define the Dupin cyclide. Figure 2 shows the two principal circles of a



**Figure 1:** The three types of Dupin cyclides. Left: ring cyclide  $0 \leq c < \mu \leq a$ . Middle: horned cyclide  $0 < \mu \leq c < a$ . Right: spindle cyclide  $0 \leq c \leq a < \mu$ .



**Figure 2:** Cut of a Dupin cyclide by its symmetry plane  $P_y$  : ( $y = 0$ ).

Dupin cyclide in plane  $P_y$ . Only a part of principal circles is useful for blending: the useful arc will be represented as a RQBC in standard form.

For example, to obtain parameters of the ring Dupin cyclide of Figure 2, we have:

$$a = \frac{O_1 O_2}{2}, \quad \mu = \frac{\rho_1 + \rho_2}{2} \quad \text{and} \quad c = \left| \frac{\rho_1 - \rho_2}{2} \right| \quad (6)$$

where  $O_i, \rho_i, i = \{1, 2\}$  are centers and radii of the principal circles.

The curvature lines of Dupin cyclides are circles, obtained with  $\theta$  or  $\psi$  constant [Pra90]. The blends with Dupin cyclides are made along lines of curvature [DMP93]: each blended surface meets the blending Dupin cyclide along lines of curvature. For more details regarding properties of Dupin cyclides, see [Pin85, CDH88, CDH89, BG92, Pra94, Pra95, She00].

### 2.3. Dupin cyclides and quadrics blending

The literature concerning the use of Dupin cyclides for the blending of quadric primitives is relatively rich, a great number of authors addressed this problem. The circular curvature lines property of Dupin cyclides facilitates the construction of the cyclide (or the portion of a cyclide) for the blending of two circular quadric primitives. Chandru et al. [CDH89] offer to use Dupin cyclides to compute variable radius blends and define an approximation strategy that can be applied in case of complex explicit construction of blends. Degen [Deg94] defines generalized cyclides as a new class of cyclides having more degrees of freedom than classical cyclides, and gives a universal representation of these surfaces and discusses the general conditions for the blending of a given surface with a generalized cyclide.

Pratt [Pra90, Pra95] shows how to make piecewise cyclide blending surfaces, and extends the symmetric cyclide blends to include asymmetric problems, the particular case of the computation of double cyclide blends between two cones is given as an example of possible blending applications. Aumann [Aum95] investigates curvature continuous blending of cones and cylinders using normal ringed surfaces which are generated, like cyclides, by a circle of variable radius sweeping through space. Allen and Dutta [AD98] present a method to generate nonsingular transition surfaces between natural quadrics using either Dupin cyclides or parabolic cyclides, necessary and sufficient conditions for the existence of the cyclide transition surface are given. Also, Allen and Dutta [AD97a, AD97b] studied a special type of blends called pure blends. Paluszny and Boehm [PB98] study the joining of quadrics and propose to work in the 4-space. So, the joining of two cones, in 3-space, is reduced to the construction of a piecewise conic, in the 4-space, that satisfies two specific conditions. Shene [She98] shows how one can use cyclides to define the blending surface of two cones. Later on, the author makes use of offset surfaces to improve and generalize this work in order to compute the blending surface of two randomly positioned half cones [She00].

We consider the blending of two quadrics differently, by computing two Dupin cyclide blends and connecting them through an intermediary cylinder that can be reduced to the intersection circle between the two Dupin cyclide blends. This intermediary cylinder is used in the same way as the use of an intermediate sphere is proposed in [DMP93] and an intermediate cone is used in [Pra95].

### 2.4. Rational Quadric Bézier Curves

Rational Quadric Bézier Curves (RQBC) are the parametric curves defined by:

$$\overrightarrow{OM}(t) = \frac{w_0 B_0(t) \overrightarrow{OP_0} + w_1 B_1(t) \overrightarrow{OP_1} + w_2 B_2(t) \overrightarrow{OP_2}}{w_0 B_0(t) + w_1 B_1(t) + w_2 B_2(t)} \quad (7)$$

where  $B_i(t)$  are Bernstein polynomials of degree 2 and parameter  $t \in [0; 1]$ . Line segment  $(P_0 P_1)$  is the tangent to the curve at point  $P_0$ , while line segment  $(P_2 P_1)$  is the tangent to the curve at point  $P_2$ .

RQBCs can be used to represent conics; taking  $w_0 = w_2 = 1$  and  $w_1 = w > 0$ , simplifies this representation and the value of  $w$  determines the nature of the represented conic: the curve is an arc of ellipse if  $0 < w < 1$ , an arc of parabola if  $w = 1$ , an arc of hyperbola if  $w > 1$ . More details on Bézier curves and surfaces can be found in Farin [Far97] and in Hoschek and Lasser [HL93].

The symmetry property of Bernstein polynomials given by:

$$\forall i \in \llbracket 0; 2 \rrbracket, \forall t \in [0; 1], B_i(t) = B_{2-i}(1-t) \quad (8)$$

is essential for the representation of circular arcs since it ensures the symmetry of geometric constructions.

### 2.5. Arc of circle determined by RQBC

In this section, we show the representation of arcs of circles using RQBC. This representation will be used to define the principal circles of the blending Dupin cyclide.

Let  $P_0$  and  $P_2$  be two given points and  $P_1$  a point of the perpendicular bisector of line segment  $(P_0P_2)$  such that  $P_1$  does not belong to this segment. There is a unique circle passing through  $P_0$  and  $P_2$  and tangent to  $(P_0P_1)$  at  $P_0$  and tangent to  $(P_2P_1)$  at  $P_2$  (Figure 3). The center  $O_0$  of this circle is characterized by the control points of the RQBC and is given by:

$$\overrightarrow{P_1O_0} = t_0 \overrightarrow{P_1I_1} \quad t_0 = \frac{P_0P_1^2}{I_1P_1 \bullet P_0P_1} \quad (9)$$

where  $I_1$  is the middle point of the line segment  $[P_0P_2]$ .

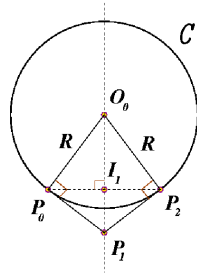


Figure 3: Modeling circular arcs using RQBCs.

### 3. Cylinder-Dupin cyclide blending

Let  $\Delta$  be the axis of the cylinder and  $C_0$  a circle on the cylinder. Let  $B_0$  and  $B_1$  be two distinct points on  $C_0$  and coplanar with  $\Delta$  (see Figure 4 left). To construct a Dupin cyclide having  $G^1$  continuity with the cylinder, we have to determine its two principal circles (Figure 4 right) such that:

- These circles are tangent to the cylinder's generatrices at points  $B_0$  and  $B_1$ .
- These circles may be defined either at the inner side or at the outer side of the cylinder.
- Centers of these circles pertain to line  $(B_0B_1)$ .

We use equation (6) to compute the cyclide parameters  $a, \mu$  and  $c$ . The trimming value for parameter  $\psi$  is  $\pi$  in the case of inner circles and 0 in the case of outer circles, while parameter  $\theta \in [0; 2\pi]$ . Figure 5 shows an example of Dupin cyclide constructed to be  $G^1$  continuous with a cylinder.

### 4. Cylinder-cone blending

To construct a  $G^1$  cyclide blend between a cone and a cylinder, we need to compute the cyclide's two principal circles  $C_1(O_1, \rho_1)$  and  $C_2(O_2, \rho_2)$ , such that each circle is tangent to both generatrices of the cone and the cylinder through their lines of curvature.

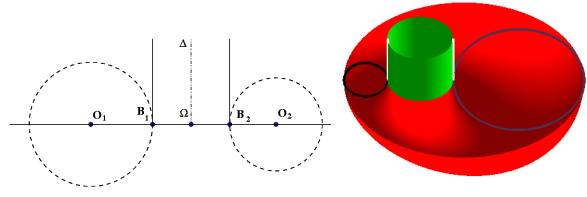


Figure 4: Cylinder-Dupin cyclide blending.

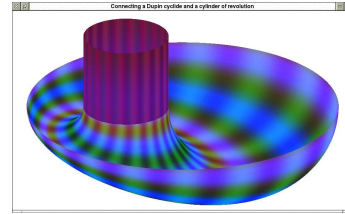


Figure 5: A cylinder-Dupin cyclide blend.

### 4.1. Computation of principal circles

Let  $\Delta_1, \Delta_2$  and  $\Delta$  be three coplanar lines.  $\Delta_1$  and  $\Delta_2$  are parallel and define the two generatrices of a cylinder of axis  $\Delta$  (see Figure 6). Let  $\Delta'_1, \Delta'_2$  and  $\Delta'$  be three coplanar lines, in the same plane as lines  $\Delta_1, \Delta_2$  and  $\Delta$ .  $\Delta'_1$  and  $\Delta'_2$  are secant and define the two generatrices of a cone of axis  $\Delta'$ . Consider the following relations:  $S = \Delta'_1 \cap \Delta'_2$ ,  $C_1 = \Delta'_1 \cap \Delta_1$  and  $C_2 = \Delta'_2 \cap \Delta_2$ .

In order to represent circles  $C_1(O_1, \rho_1)$  and  $C_2(O_2, \rho_2)$  using rational quadric Bézier curves we need to compute four control points  $B_1 \in \Delta_1, A_1 \in \Delta'_1, B_2 \in \Delta_2$  and  $A_2 \in \Delta'_2$ , such that  $A_1C_1 = B_1C_1$  and  $A_2C_2 = B_2C_2$ . We note that, by definition, points  $B_1$  and  $B_2$  are symmetrical according to  $\Delta$ , while points  $A_1$  and  $A_2$  are symmetrical according to  $\Delta'$ .

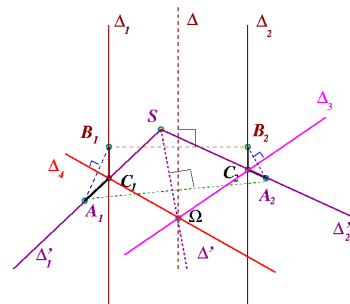


Figure 6: Construction used for cylinder-cone blending.

A necessary condition on the computation of these points is that lines  $\Delta, \Delta_4, \Delta_3$  and  $\Delta'$  must be concurrent at point  $\Omega$ . The computation can be done as follows:

- Compute  $\Delta_4$  the bisector of  $(\widehat{\Delta'_1; \Delta_1})$  and  $\Delta_3$  the bisector of  $(\widehat{\Delta_2; \Delta'_2})$  (see Figure 6).
- Define an application  $s_D(p)$  that gives point  $p'$  symmetrical to a given point  $p$  according to an axis  $D$ .
- Choose any point  $A_1 \in \Delta'_1$ ,
- Other points are the result of the following transformations:  $B_1 = s_{\Delta_4}(A_1)$ ,  $B_2 = s_{\Delta_3}(B_1)$  and  $A_2 = s_{\Delta_3}(B_2)$ .
- $A_1 = s_{\Delta'_1}(A_2)$  iff  $s_{\Delta'_1} \circ s_{\Delta'_3} \circ s_{\Delta'_2} \circ s_{\Delta'_4}$  is the identity application (a rotation of angle 0 centered at  $\Omega$ ).

The Dupin cyclide principal circles  $C_1(O_1, \rho_1)$  and  $C_2(O_2, \rho_2)$  can now be obtained using formula (9).

#### 4.2. Examples

From the principal circles, it is easy to construct the Dupin cyclide blend between the cylinder and the cone. We have only to determine value  $\psi_1$  delimiting the Dupin cyclide from the cone side, which means: find the value  $\psi_1$  such that  $\Gamma(\theta; \psi_1)$  of equation (3) is a circle on the cone.

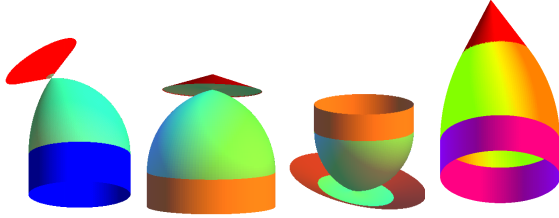


Figure 7: Cone-cylinder spindle Dupin cyclide blends.

Figures 7 and 8 show a set of examples of constructed Dupin cyclide blends. Blends of Figure 7 are trimmed spindle Dupin cyclides, while those of Figure 8 are trimmed ring Dupin cyclides.

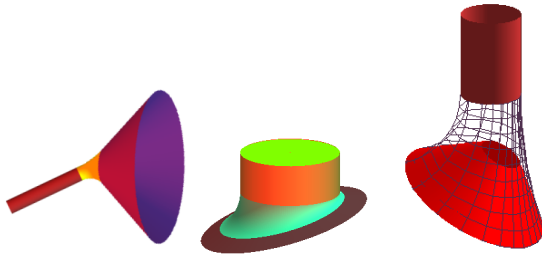


Figure 8: Cone-cylinder ring Dupin cyclide blends.

#### 5. Cylinder-sphere blending

In this case, we regard the computation of the principal circles of the Dupin cyclide as a double computation of a circle tangent to a line segment  $\Delta$  and a circle  $C$ , where  $\Delta$  is one of

the generatrices of the cylinder and  $C$  is one of the main circles of the sphere. In this construction we consider that the main circle of the sphere and the cylinder axis are coplanar.

#### 5.1. Computation of the principal circles

Let  $\Delta$  be a line segment and  $C$  a circle, we look for a circle tangent to  $\Delta$  and to  $C$  as well, that amounts to computing the three control points  $A, B$  and  $C$  of the rational quadratic Bézier curve that models the circle, points  $B$  and  $C$  pertain to  $\Delta$ , while point  $A$  pertains to  $C$  such that  $AB = AC$  and  $(AC) \perp (A\Omega)$  (Figure 9). Algorithm (1) details the computation of these points.

**Algorithm 1** Obtaining Bézier control points for a  $G^1$ -continuous circular arc blend between a straight line and a circle

Given: A straight line  $\Delta$ , a circle  $C$ , of radius  $r$  and center  $\Omega$ , and a point  $B \in \Delta$ .

1. Build  $F$  the image of  $\Omega$  by the orthogonal projection of axis  $\Delta$
2.  $G$  is one of the intersection between  $(F\Omega)$  and  $C$
3.  $A$  is in the intersection between  $(GB)$  and  $C$
4.  $C$  is the intersection between  $\Delta$  and the tangent to  $C$  at point  $A$

Result :  $AC = BC$ ,  $C \in \Delta$ ,  $A \in C$  and  $(A\Omega) \perp (AC)$ .

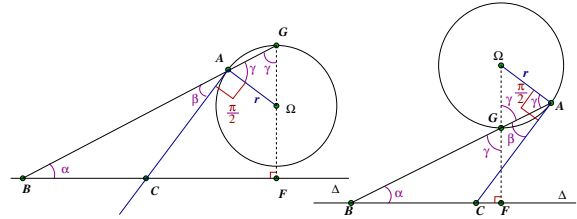


Figure 9: Circle tangent to a circle and a line segment.

To prove the computation of points  $A, B$  and  $C$ , let us consider the two possible cases depicted in Figure 9.

On Figure 9 left, we have  $\widehat{BAG} = 180^\circ$  and  $(CA) \perp (A\Omega)$  so  $\widehat{BAC} + \widehat{\Omega AG} = 90^\circ$ . Triangle  $BFG$  is right-angled and  $A \in [BG]$  so  $\widehat{FBA} + \widehat{AG\Omega} = 90^\circ$ .  $\Omega AG$  is an isosceles triangle in  $\Omega$  so  $\widehat{\Omega AG} = \widehat{AG\Omega} = \gamma$ , which implies  $\alpha = \widehat{FBA} = \widehat{BAC} = \beta$  and then  $BAC$  is an isosceles triangle in  $C$ , so  $BC = CA$ .

On Figure 9 right, we have  $\widehat{AG\Omega} = \widehat{BGF}$  because these angles are opposite by the vertex.  $BFG$  is a right-angled triangle so  $\alpha + \gamma = 90^\circ$ .  $\Omega AG$  is an isosceles triangle in  $\Omega$  so  $\widehat{\Omega AG} = \widehat{AG\Omega} = \gamma$ .  $\widehat{\Omega AC} = 90^\circ$  and  $\widehat{\Omega AC} = \widehat{\Omega AB} + \widehat{BAC} = \beta + \gamma$  so  $\beta + \gamma = 90^\circ$ . As  $\alpha + \gamma = 90^\circ$ , we have  $\alpha = \beta$  so  $BAC$  is an isosceles triangle in  $C$  and then  $BC = CA$ .

Computing principal circles of the Dupin cyclide blend is now straightforward (Figure 10), we only have to apply the above computations of  $A, B$  and  $C$  twice with the two generatrices of the cylinder and the main circles of the sphere to obtain the necessary control points  $A_1, C_1$  et  $B_1$  for the first circle and  $A_2, C_2$  et  $B_2$  for the second one. Constructions of Figure 10 are made in the plane determined by the axes of the cylinder and the center of the main circle of the sphere. Algorithm (2) indicates how these computations are done.

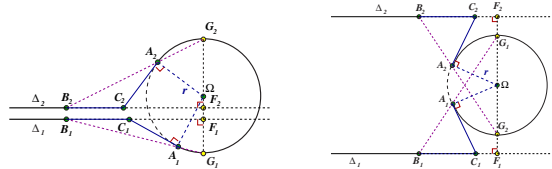


Figure 10: Constructions for cylinder-sphere blending.

**Algorithm 2** Principal circles of Dupin cyclide blends between a cylinder and a sphere.

Given: A cylinder with axis  $\Delta$  and two generatrices  $\Delta_1$  and  $\Delta_2$ , a sphere with a main circle  $C(\Omega, r)$ , and two points  $B_1 \in \Delta_1$  and  $B_2 \in \Delta_2$ , such that  $[B_1 B_2]$  is perpendicular to  $\Delta$ .

1. The axis  $\Delta$  determines two half-planes,  $P_1$  and  $P_2$  such that  $\Delta_1 \in P_1$  and  $\Delta_2 \in P_2$
2. For  $i \in \{1, 2\}$ , find points  $G_1 \in P_1$  and  $G_2 \in P_{3-i}$  and then compute points  $F_i, A_i, C_i$  in  $P_i$

Result:  $A_1, C_1$  and  $B_1$  are control points of the blending circle on one side, and  $A_2, C_2$  and  $B_2$  are control points of the blending circle on the other side. Dupin cyclide blend can now be constructed taking these circles as principal circles.

### 5.2. Examples

From the principal circles we proceed in the same way as for the examples of subsection 4.2 to compute Dupin cyclide blends (construct and trim the surface).

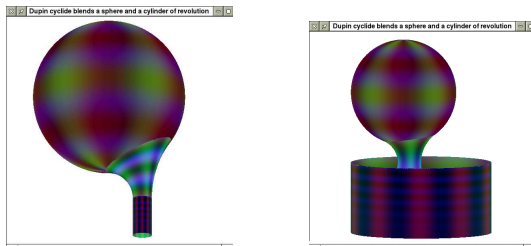


Figure 11: Sphere-cylinder blending using ring Dupin cyclides.

Figures 11 and 12 show four examples of blends between a cylinder and a sphere by ring and spindle Dupin cyclides.

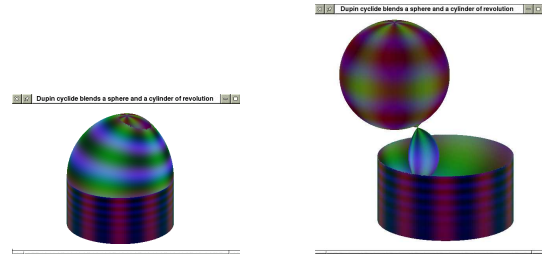


Figure 12: Sphere-cylinder blending using spindle Dupin cyclides.

### 6. Cone-sphere blending

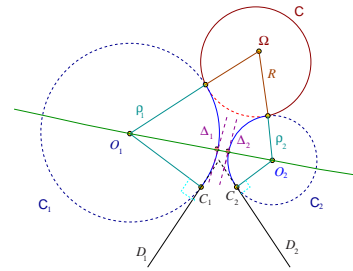


Figure 13: Construction used for cone-sphere blending. The cylinder defined by  $\Delta_1$  and  $\Delta_2$  is introduced to simplify the operation.

The use of the blending methods developed in sections 4 and 5 allows an easy computing of cone-sphere blends. We decompose this operation into two sub-operations: cone-cylinder and cylinder-sphere. The same cylinder is used for both sub-operations (see Figure 13) and is generally reduced to a circle (the common intersection of the two Dupin cyclide blends).

This novel method of cone-sphere blends computation is applied on the examples of figure 14, shown results give an idea of the different possible combinations one can do. Each one of the cone-sphere blending algorithms proposed in [Pra90, DMP93, GFN03] give a blend composed of one single cyclide as we can see on the top-left sub-figure of figure 14, parameters of the resulting cyclide are:  $a \simeq 10.23$ ,  $\mu \simeq 7.04$ ,  $c \simeq -2.42$ . The blend obtained using our algorithm may be composed of either two cyclides or two cyclides and a cylinder. In the first case, the cyclides have  $G^1$  continuity through their circle of intersection (see the top-right sub-figure of figure 14). Parameters of the cyclide on the cone side are:  $a \simeq 14.1$ ,  $\mu \simeq 11.79$ ,  $c \simeq -6.09$ , while parameters of the cyclide on the sphere side are:  $a \simeq 8.09$ ,  $\mu \simeq 5.77$ ,  $c \simeq -2.58$ . In the other case, each cyclide has a  $G^1$  continuity with the cylinder. In the bottom sub-figures of figure 14, we construct the cylinder by changing only the parameters of the cyclide on the sphere side, we have:  $a \simeq 7.24$ ,

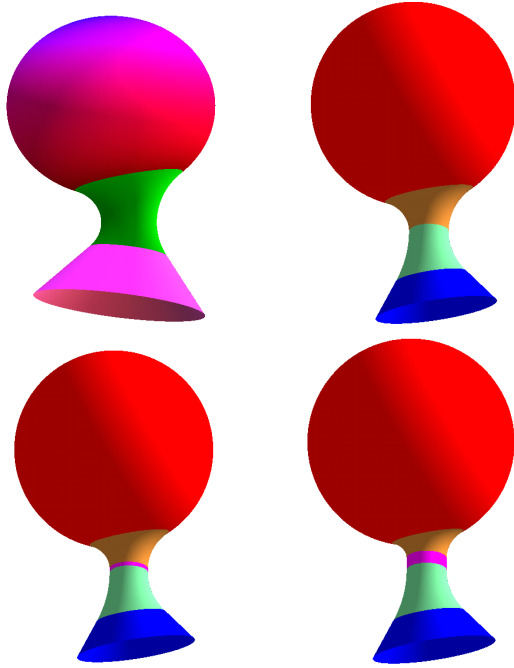


Figure 14: Examples of cone-sphere blending results.

$\mu \simeq 4.93$ ,  $c \simeq -2.31$  when the cylinder height is 0.5, and  $a \simeq 5.63$ ,  $\mu \simeq 3.31$ ,  $c \simeq -1.8$  when the cylinder height is 1.5. M. Pratt has also proposed examples of such double-cyclide blends in [Pra95].

Figure 15 shows another application of our algorithm where a cylinder together with both ring and spindle cyclides is used to construct composed blends. In the top sub-figures, the blend is composed of two ring cyclides and a cylinder. In the bottom sub-figures, the blend is composed of a ring cyclide (on the cone side), a cylinder and a spindle cyclide (on the sphere side).

## 7. Modeling Eurographics' Hugo

In this section we aim at showing how we can combine a set of quadric primitives using Dupin cyclide blends to model the Eurographics' Hugo. This object is available for download, in different formats, on the web site of Eurographics 2004 conference at <http://eg04.inrialpes.fr/Download/hugo.html>. For this modeling, quadric primitives are used to represent Hugo's parts (head, body, hands ...), and cyclides allows  $G^1$  continuous blending of these parts. Also a comparison, in terms of size and memory consumption, between our designed shape and the mesh format of Hugo is given. Our model of Hugo is shown on figure 16.

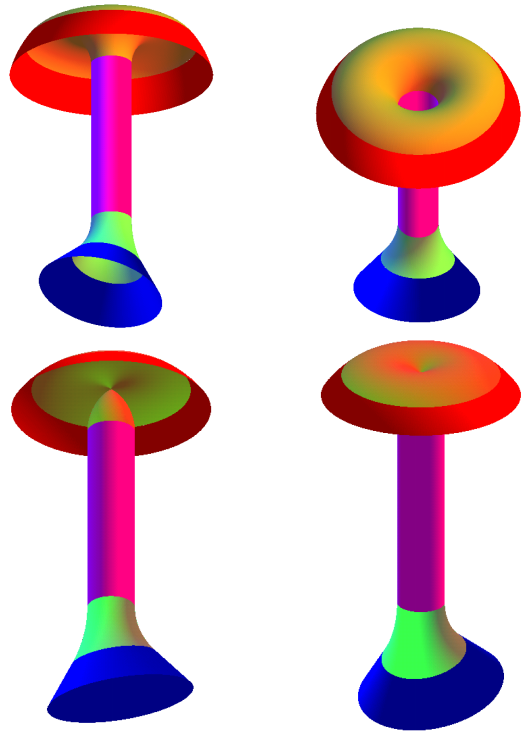


Figure 15: Examples of cone-sphere blends combining both ring and spindle Dupin cyclides.

### 7.1. Used quadric primitives

Table 1 summarizes the primitives used to construct a surface representation of Hugo. For each primitive we used, column two gives the number of occurrences, the shown value is computed after all the necessary geometric transformation (translations, rotations), column three gives the memory usage, in terms of floating numbers, while column four gives parts of Hugo where the primitive is used. From this table, one can see that to represent the whole shape we have used 87 primitives: 14 spheres, 5 cones, 14 ellipsoids, 24 cylinders and 30 Dupin cyclides. The total size of this model is  $14 \times 4 + 5 \times 7 + 14 \times 5 + 24 \times 7 + 30 \times 11 = 659$  floating numbers, this is more than optimal compared to the original model which contains 8388 faces ie. around 15492 floating numbers. Another advantage of this surface representation of Hugo is the ease to animate the object. For example, to put up Hugo's hand we have to change only one value (the rotation angle between the arm and the shoulder), OpenGL transformations do the necessary computations and visualize Hugo at the new position.

Although most blends are  $G^1$  continuous, some blends such as: the head cylinder-{ear, eye, nose, mouth}, the wrist sphere-fingers cylinders, shoulder sphere-body cone, are only  $G^0$  continuous because these connections are made

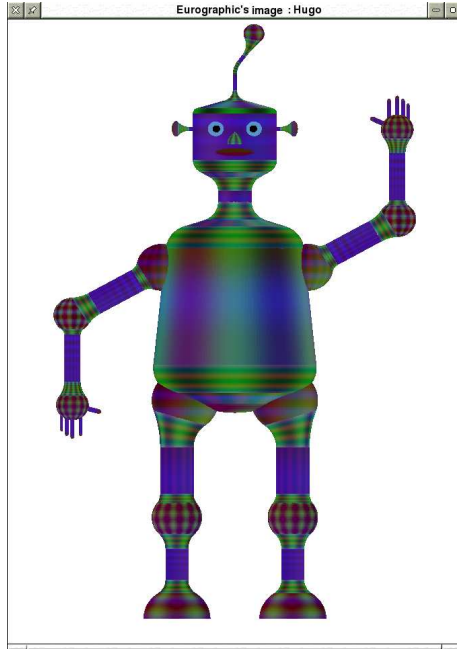


Figure 16: Eurographics Hugo represented by a set of quadrics blended by Dupin cyclides.

Primitive	Number	Size	Where
sphere	14	4	bonnet, 2 wrists, 2 elbows, 2 clavicles, 2 hips, 2 knees, buttocks, 2 feet
cone	5	7	cranium, chin, pelvis, body, shoulder
ellipsoid	14	5	2 ears, 2 eyes, 10 finger tips
cylinder	24	7	2 bonnet stems, 2 arms, 2 ear stems, 2 thighs, 2 front arms, 2 calves, 10 fingers, face, neck
cyclide	30	11	Different blends

Table 1: Quadric and cyclide primitives used to model Hugo.

with the cylindrical side of primitives being blended. Next subsection details the modeling of the Hugo’s head and shows the construction of all blends.

## 7.2. Details on Hugo’s head

For the modeling of Hugo’s head, axes of cylinders and cones are in the same plane  $P_y$ . Constructions shown on figures 17 and 18 are made in this plane.

In figure 17, line segments  $[A_0A_1]$  and  $[B_0B_1]$  are on the two generatrices of the neck cylinder. Line segments  $[A_3A_4]$  and  $[B_3B_4]$  are on the two generatrices of the chin cone. We define points  $A_2 = [A_0A_1] \cap [A_3A_4]$  and  $B_2 = [B_0B_1] \cap [B_3B_4]$  such that  $A_3A_2 = A_2A_1$  and  $B_3B_2 = B_2B_1$ . Using the RQBCs representation of circular arcs we define a circular arc passing through  $A_1$  and  $A_3$  and centered at  $O_0$ , so this arc is tangent to one generatrix of the cylinder as well as to one generatrix of the cone. In the same way we define another arc passing through  $B_1$  and  $B_3$  and centered at  $I_0$ . From these two arcs we compute the principal circles and then the parameters of a Dupin cyclide  $G^1$  continuous blend between the neck cylinder and the chin cone.

In figure 18, we blend the two cylinders of the bonnet such that points  $A_{13}, A_{14}, A_{15}$  and  $B_{13}, B_{14}, B_{15}$  constitute the control point of the RQBC representing the needed circular arcs. The Dupin cyclide symmetry plane  $P_z$  is used to determine its parameters.

To model the head with the bonnet, see figure 19, we have used:

- 2 ellipsoids for the ear and the eye
- 6 cylinders for the neck, the face, the 2 stems of the bonnet and the 2 stems of the ear
- 1 sphere for the bonnet bobble
- 2 cones for the cranium and the chin
- 2 horned Dupin cyclides for the mouth
- 1 spindle Dupin cyclide for the nose
- 8 ring Dupin cyclides as  $G^1$  blends for: bonnet bobble-bonnet upper stem, 2 stems of the bonnet, bonnet lower stem-cranium, cranium-face, face-chin, chin-neck, and 2 ear-ear stem.

In the following four particular cases where the connection is made with the cylindrical side of one of the two primitives being blended, we have chosen to make  $G^0$  blends: ear cylinder-face cylinder, eye ellipsoid-face cylinder, mouth horned Dupin cyclide-face cylinder and nose spindle Dupin cyclide-face cylinder. All other blends are  $G^1$  continuous.

## 8. Conclusion

This paper shows how Dupin cyclides can be used as blends between quadric primitives. We regard the blending of two quadric primitives A and B as two complementary blending operations: primitive A-cylinder and cylinder-primitive B; Two Dupin cyclides and a cylinder are then defined for each blending operation. In general the cylinder is not useful and may be reduced to a simple circle. We use the rational quadratic Bézier curves representation of circular arcs to model the principal circles. These circles are constructed to

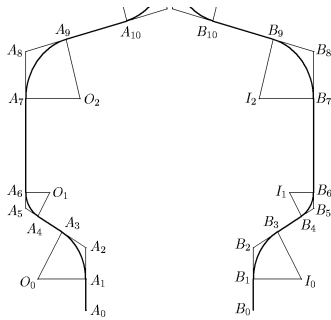


Figure 17: Construction of Hugo's head.

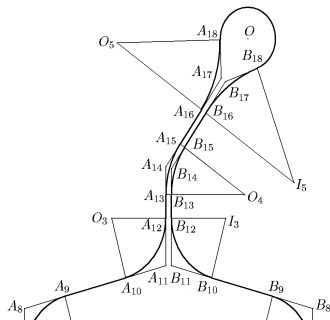


Figure 18: Construction of Hugo's bonnet.

be tangent to the primitives being blended, so that the Dupin cyclide based on these circles constitutes a  $G^1$  blend between the primitives. The proposed algorithm is very easy to apply. The user only has to give a point on the cone to define a cylinder-cone blend, one point on the cylinder and another one on the sphere to define a cylinder-sphere blend. A complete shape design example is presented to illustrate our contributions.

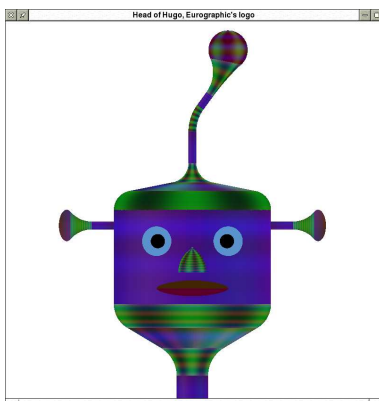


Figure 19: Zoom on Hugo's head.

The use of supercyclides for the blending of elliptical quadric primitives is now under consideration.

### Acknowledgments

The authors thank Bruno Ingrao from Blaise Pascal University, Clermont-Ferrand, France, for his help and comments regarding some geometric problems related to this work, also our thanks are to l'Ecole des Applications Militaires de l'Energie Atomique of Cherbourg, France, for the financial support.

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