GEOMETRIC OPTIMAL CONTROL OF THE ATMOSPHERIC ARC FOR A SPACE SHUTTLE

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Abstract. We give preliminary remarks concerning the optimal control of the atmospheric arc for a space shuttle (earth re-entry or Mars sample return project). The system governing the trajectories is 6–dimensional, the control is the bank angle, the cost–integrand is the thermal flux and we have state constraints on the thermal flux and the normal acceleration. Our study is geometric and founded on the analysis of the solutions of a minimum principle and direct evaluation of the small–time reachable set for the problem taking into account the state constraints.

1. Introduction

The objective of this article is to make a preliminary analysis of the optimal control of the atmospheric arc for a space shuttle where the cost is the total thermal flux. The control is the bank angle (the angle of attack being hold fixed) and we have state constraints on the thermal flux and the normal acceleration. A pure numerical approach to the problem is presented in [2] where the analysis is also simplified because the terminal condition is relaxed to a condition on the modulus of the speed. Our aim is to analyze the problem with fixed end-point conditions which leads to a complex control law due to the number of switchings (or the number of rotations) we need to match the boundary conditions.

This article is only a first step in the analysis in order to introduce the geometric tools to handle the problem and the necessary optimality conditions. In particular we shall restrict our computations to a 3 dimensional subsystem where the state variables are the modulus of the velocity, the altitude and the flight path angle. Also we shall localize the analysis to a small neighborhood of any point in the flight domain. This will allows to give local bounds to the number of switchings. It must be completed by numerical simulations to get a global bound.

Our approach is geometric and use necessary optimality conditions and direct evaluation of the small time reachable set in the spirit of [11] but using normal forms as in [2] where the constraints are taken into account. It is well illustrated by the following planar example. Consider the time optimal control problem for the system $\dot{q} = X(q) + uY(q)$, $q = (x, y)$, $|u| \leq 1$. Let γ_+ (resp. γ_-) be an arc

Date: November 6, 2000.

1991 *Mathematics Subject Classification.* AMS Classif. 49K15,70H20.

Key words and phrases. Optimal control, Nonlinear systems with state constraints, Minimum principle, Reachable set, Space shuttle re–entry problem and Mars sample return project.

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corresponding to $u = +1$ (resp. $u = -1$) and denote $\gamma_1\gamma_2$ an arc γ_1 followed by γ_2. Take a generic point q_0, then the small time reachable set starting from q_0 is a cone bounded by arcs γ_+ and γ_- and each optimal trajectory is of the form $\gamma_+\gamma_-$ or $\gamma_-\gamma_+$, see figure 1; moreover along a trajectory the time can be measured using Miele's form: $\omega = p\,dq$ where p is given by $\langle p, X\rangle = 1$, $\langle p, Y\rangle = 0$.

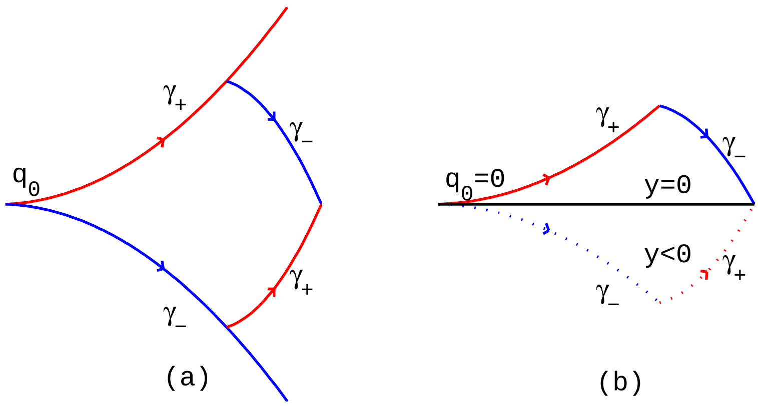

FIGURE 1. Reachable set with and without constraints

Assume $q_0 = 0$ and the trajectories constrained to the domain $C : y \geq 0$. Let $\gamma_b(t)$, $t \in [0, T]$ be a boundary arc starting from $q_0 = 0$ and contained in the frontier $y = 0$; assume that the corresponding control u_b is admissible and not saturating. Let $B = \gamma_b(t)$, $T > 0$ small enough. Consider the arcs $\gamma_+\gamma_-$ and $\gamma_-\gamma_+$ joining 0 to B, one is time minimal (and the other is time maximal) for the problem without state constraint, and we have two possibilities for the constrained problem, see figure 1,(b). Assume it is $\gamma_+\gamma_-$, then if it is contained in $y \geq 0$, it is admissible and the boundary arc is not optimal, the optimal synthesis near q_0 for the constrained system being $\gamma_+\gamma_-$. If $\gamma_+\gamma_-$ is not contained in $y \geq 0$ the boundary arc is time optimal and the optimal synthesis is $\gamma_+\gamma_b\gamma_-$. The analysis can be carried out in full details using the model $\dot{x} = 1 + ay$, $\dot{y} = c + u$ and not the Miele's form ω defined only for planar systems.

A major problem when analyzing optimal control problems with state constraints is to derive necessary optimality conditions. Indeed the constraints can be penalized in the cost in several manners and this leads to introduce the concept of order of the constraints. Also it is the basic concept to construct normal forms and evaluate the reachable sets for the system with the constraints. We shall formulate a minimum principle due to [8, 12], adapted to analyze the optimal trajectories for the space shuttle. It concerns single input control systems and we need regularity assumptions. It is much more precise than the general minimum principle of [12], where an optimal trajectory is the projection of a trajectory in cotangent bundle depending of a measure supported by the constraints.

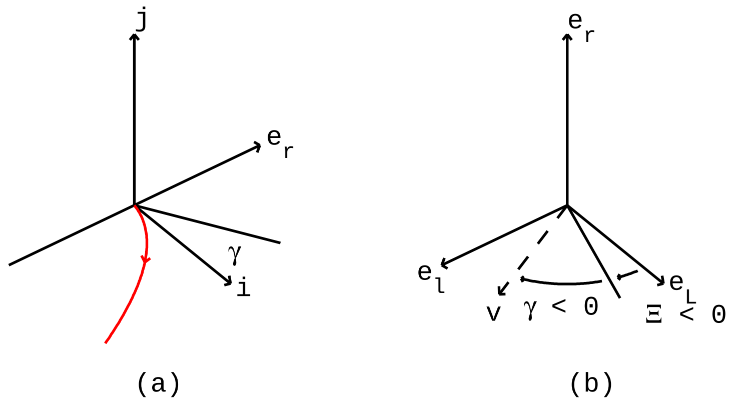

FIGURE 2. Moving frames: flight path angle and azimuth

2. The model

The problem is to control the atmospheric arc nearby a planet which can be the Earth (re–entry problem) or Mars (sample return project). In both cases the equations are the same, except for constants related to the planet (radius, mass, angular velocity, atmosphere). In our computations we shall assume that the planet is the Earth. In order to modelize the problem, we use the laws of classical mechanics, a model of the gravitational force, a model of atmosphere and a model of the aerodynamic force which decomposes into a drag force and a lift force.

The equations are simplified by choices of orthonormal moving frames that we explain below.

2.1. **Moving frames.** We denote by $E = (e_1, e_2, e_3)$ a standard Galilean frame whose origin O is the center of the Earth and let $R_1 = (I, J, K)$ be a rotating frame centered at 0 where K is the axis N–S of rotation of the Earth, the angular velocity being Ω and I is chosen to intersect Greenwich meridian.

Let R be the Earth radius and let G be the center of mass of the shuttle. We denote by $R_1' = (e_r, e_l, e_L)$ the frame associated to spherical coordinates of $G = (r, l, L)$, $r \geq R$ being the distance OG and l, L being respectively the longitude and latitude.

We introduce the following moving frame $R_2 = (i, j, k)$ whose center is G. Let $\zeta : t \to (x(t), y(t), z(t))$ be the trajectory of G measured in the frame R_1 and let $\overrightarrow{v}$ be the relative speed $v = \dot{x}I + \dot{y}J + \dot{z}K$. To define $\overrightarrow{i}$, we set $\overrightarrow{v} = |v|\,\overrightarrow{i}$. The vector j is a vector in the plane (i, e_r), j is perpendicular to i and oriented by $j.e_r > 0$. We take $k = i \wedge j$. The vector i is parametrized in the frame $R_1' = (e_r, e_l, e_L)$ by two angles:

- γ: flight path angle
- Ξ: azimuth

defined on figure 2.

2.2. **Model of the forces.** For the atmospheric arc we assume the following

Assumption 1. *There is no thrust: the shuttle is a glider.*

Assumption 2. *The speed of the atmosphere is the speed of the Earth, i.e. the relative speed of the shuttle with respect to the atmosphere is the speed* $\overrightarrow{v}$.

We must consider two types of forces acting on the shuttle.

- Gravitational force. We assume that the Earth is spherical so that the gravitational force is oriented along e_r.It is written in the moving frame R_2

$$\overrightarrow{P} = -mg\left(i\sin\gamma + j\cos\gamma\right)$$

  where $g = \frac{\mu_e}{r^2}$.
- Aerodynamic force. The effect of the atmosphere on the shuttle is on aerodynamic force which decomposes into
  - A drag force colinear to the speed $\overrightarrow{v}$ and of the form

$$\overrightarrow{T} = -\left(\frac{1}{2}\rho S C_D v^2\right) i$$

  - A lift force perpendicular to $\overrightarrow{v}$ and given by

$$\overrightarrow{P}_T = \frac{1}{2}\rho S C_L v^2\left(j\cos\mu + k\sin\mu\right)$$

    and μ is called the bank angle, where $\rho = \rho(r)$ is the atmospheric density, S is a constant and C_D, C_L are respectively the drag and lift coefficient.

Assumption 3. *Both coefficients C_D and C_L are depending upon the angle of attack α which parametrized the orientation of the speed v with respect to the normal of an element of area of the shuttle. We assume that for the atmospheric arc the angle of attack is kept constant. This is very restrictive but it is worth to point out that in the numerical simulations of* [2] *where α is a control, in the optimal solution it is a constant.*

Hence the only control is the angle of bank μ.

2.3. **System equations.** The atmospheric arc is governed by the following system

$$\frac{dr}{dt} = v \sin(\gamma) \tag{2.1a}$$

$$\frac{dv}{dt} = -g \sin(\gamma) - \frac{1}{2}\rho\frac{S\,C_D}{m}v^2 + \Omega^2 r \cos L \left(\sin\gamma \cos L - \cos\gamma \sin L \cos \Xi\right) \tag{2.1b}$$

$$\frac{d\gamma}{dt} = \cos(\gamma)\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{S\,C_L}{m}v\cos(\mu) + 2\Omega\cos L \sin\Xi \tag{2.1c}$$

$$+\,\Omega^2\frac{r\cos L}{v}\left(\cos\gamma\cos L + \sin\gamma \sin L \cos\Xi\right) \tag{2.1d}$$

$$\frac{dL}{dt} = \frac{v}{r}\cos\gamma\cos\Xi \tag{2.1e}$$

$$\frac{dl}{dt} = -\frac{v}{r}\frac{\cos\gamma\sin\Xi}{\cos L} \tag{2.1f}$$

$$\frac{d\Xi}{dt} = \frac{1}{2}\rho\frac{SC_L}{m}\sin\mu\frac{v}{\cos\gamma} + \frac{v}{r}\cos\gamma\tan L\sin\Xi \tag{2.1g}$$

$$+\,2\Omega\left(\sin L - \tan\gamma\cos L\cos\Xi\right) + \Omega^2\frac{r}{v}\frac{\sin L\cos L\sin\Xi}{\cos\gamma} \tag{2.1h}$$

where the control is the bank angle μ and the state space is $q = (r, v, \gamma, L, l, \Xi)$

2.4. **Atmospheric model.** Atmospheric density is tabulated for Earth, Mars and Venus and we take an exponential model

$$\rho = \rho_0 \mathrm{e}^{-\beta r}$$

3. The control problem

3.1. **Control and control bounds.** The control can be either μ or $\dot{\mu}$. In the first case we can have the following bounds: $\mu \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\mu \in [-\pi, \pi]$. We set $u_1 = \cos\mu$ and u_1 is a direct control on the flight path angle γ. We let $u_2 = \sin\mu$ and u_2 control the azimuth, the sign of u_2 allows the glider to turn left or right.

3.2. **State constraints.** There are several state constraints but in the first step of our analysis we shall consider two constraints:

- Constraint on the thermal flux

$$\varphi = C_q\sqrt{\rho}v^3 \leq \varphi^{\max} \tag{3.1}$$

  where C_q is a given constant

- Constraint on the normal acceleration

$$\gamma_n = \gamma_{n_0}(\alpha)\,\rho v^2 \leq \gamma_n^{\max} \tag{3.2}$$

3.3. **Optimal cost.** Several choices are allowed and we make the analysis for

$$J(\mu) = \int_0^T C_q\sqrt{\rho}v^3 dt \tag{3.3}$$

which represents the total thermal flux, the duration T of the atmospheric arc being not fixed. We introduce the differential equation

$$\frac{d\widetilde{q}_0}{dt} = C_q\sqrt{\rho}v^3 \tag{3.4}$$

with $\widetilde{q}_0(0) = 0$.

3.4. **Boundary conditions.** The transfer time T is free and we have two choices for the boundary conditions:

- Fixed boundary conditions at $t = 0$ and $t = T$ for $q = (r, v, \gamma, L, l, \Xi)$.
- $\gamma \in [\gamma_{\min}, \gamma_{\max}]$ at $t = 0$ with the constraint that a preliminary maneuver on the Keplerian arc allows this possibility.

3.5. **State domain for the atmospheric arc.** The flight domain D for the Earth re–entry of the shuttle is the following:

- Altitude: $h = r - R \in [40\,\mathrm{km}, 120\,\mathrm{km}]$
- Velocity amplitude $v \in [2000\,\mathrm{m}\,/\,\mathrm{s}, 8000\,\mathrm{m}\,/\,\mathrm{s}]$
- The flight path angle domain is $0 > \gamma > -15\,^{\circ}$.

Assumption 4. *(controllability assumption) The Earth angular velocity Ω is small and hence*

$$\frac{d\gamma}{dt} \cong \cos(\gamma) \left(-\frac{g}{v} + \frac{v}{r} \right) + \frac{1}{2} \rho \frac{S\, C_L}{m} v \cos(\mu) \tag{3.5}$$

We shall denote by D_c the subset of D where the lift force can at each point compensated the gravitational force that is

$$\frac{1}{2} \rho \frac{S\, C_L}{m} v > \frac{g}{v}$$

and (3.5) is feedback linearizable in the domain.

4. The minimal principle without state constraints – Extremal curves

4.1. **Problem statement and notations.** Let the single–input control system

$$\dot{q} = F(q, u) \tag{4.1}$$

and a cost to be minimized of the form

$$J(u) = \int_0^T \varphi(q)\, dt \tag{4.2}$$

where the transfer time T is free and φ is not depending upon u. The set of admissible controls is the set $\mathcal{U}$ of measurable mappings $u : [0, T] \to U$. The state domain is a subset of $\mathbb{R}^n$ with the state constraints:

- Constraint on the thermal flux

$$c_1(q) = \varphi(q) \le \alpha_1 \tag{4.3}$$

- Constraint on the normal acceleration

$$c_2(q) = \gamma_n(q) \le \alpha_2 \tag{4.4}$$

The boundary conditions are of the form:

- $q(0) = q_0$ and $q(T) = q_1$ fixed

or

- if $q = (r, v, \gamma, L, l, \Xi)$ then $\gamma(0) \in [\gamma_1, \gamma_2]$, $\gamma_1 < \gamma_2 < 0$.

We denote by $R(q_0, t)$ the reachable set at time $t > 0$ fixed and $R(q_0) = \bigcup_{t \text{ small enough}} R(q_0, t)$ the small time reachable set.

4.2. **Minimum principle.** We recall the minimum principle [13] which allows to parametrize the boundaries of the reachable sets [11].

We introduce the Hamiltonian

$$H(q,p,u) = \langle p, F(q,u)\rangle + \widetilde{p}_0\varphi(q)$$

where $q = (r, v, \gamma, L, l, \Xi)$ and $p = (p_r, p_v, p_\gamma, p_L, p_l, p_\Xi)$ is the adjoint vector and $\widetilde{p}_0$ is a constant such that $(p, \widetilde{p}_0) \neq 0$.

Definition 1. *If $\widetilde{p}_0 \neq 0$ we are in the normal case and if $\widetilde{p}_0 = 0$ we are in the abnormal case.*

Definition 2. *We call extremal a triplet (q,p,u) solution of the minimum principle*

$$\dot{q} = F(q,u) = \frac{\partial H}{\partial p} \tag{4.5}$$

$$\dot{p} = -p\frac{\partial F}{\partial q} - \widetilde{p}_0\frac{\partial \varphi}{\partial q} = -\frac{\partial H}{\partial q} \tag{4.6}$$

$$H(q,p,u) = \min_{w\in\mathcal{U}} H(q,p,w) \tag{4.7}$$

Proposition 1. *An optimal solution for the problem without state constraint is a projection on the state space of an extremal solution. Moreover $\widetilde{p}_0 \geq 0$. Since the transfer time T is free it is exceptional, that is $H = 0$. If moreover γ is free at $t = 0$, the adjoint vector p satisfy the transversality condition*

$$p_\gamma(0) = 0 \text{ if } \gamma(0) \in \left]\gamma_1, \gamma_2\right[\tag{4.8}$$

4.3. **Definition of subsystem (I).** Observe that Ω is small with respect to the velocity of the shuttle. Hence if we neglect the transport terms $O\left(\Omega^2\right)$ and the Coriolis terms $O(\Omega)$ our system can be decomposed with $q_1 = (r, v, \gamma)$ and $q_2 = (L, l, \Xi)$ into

$$\dot{q}_1 = F_1(q_1, u_1)$$
$$\dot{q}_2 = F_2(q, u_2)$$

where $u_1 = \cos\mu$, $u_2 = \sin\mu$, $u = (u_1, u_2)$ and

$$U = \left\{u_1^2 + u_2^2 = 1 \text{ and } u_1 \geq 0 \text{ if } \mu \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$$

The adjoint system (4.6) is the decomposed into

$$\begin{pmatrix} \dot{p}_1 & \dot{p}_2 & 0 \end{pmatrix} = -\begin{pmatrix} p_1 & p_2 & \widetilde{p}_0 \end{pmatrix}\begin{pmatrix} \frac{\partial F_1}{\partial q_1} & 0 & 0 \\ \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2} & 0 \\ \frac{\partial \varphi}{\partial q_1} & 0 & 0 \end{pmatrix}$$

If we relax the end–point condition on $q_2 = (L, l, \Xi)$ we obtain using the transversality condition $p_2(T) = 0$ and hence $p_2(t) \equiv 0$. The analysis of extremals reduces to the analysis of the solutions of

$$\dot{q}_1 = F_1(q_1, u_1)$$
$$\dot{p}_1 = -p_1\frac{\partial F_1}{\partial q_1} - \widetilde{p}_0\frac{\partial \varphi}{\partial q_1}$$

It is associated to the optimal control of system (I) given below:

$$(\mathrm{I})\quad \left\{\begin{array}{lcl} \dfrac{dr}{dt} & = & v\sin(\gamma) \\ \dfrac{dv}{dt} & = & -g\sin(\gamma)-\dfrac{1}{2}\rho\dfrac{S\,C_D}{m}v^2 \\ \dfrac{d\gamma}{dt} & = & \cos(\gamma)\left(-\dfrac{g}{v}+\dfrac{v}{r}\right)+\dfrac{1}{2}\rho\dfrac{S\,C_L}{m}v\cos(\mu) \end{array}\right.$$

Note that $q_1=(r,v,\gamma)$ appears only in the state–constraints. We shall concentrate in a first step our analysis on the subsystem (I). It is related to the numerical simulation of [2].

4.4. **Analysis of extremals of system (I).**

4.4.1. *Problem reduction and definitions.* Consider a single input affine system

$$\dot{q}=X+uY,\ |u|\le 1 \tag{4.9}$$

and a cost to be minimized of the form

$$J(u)=\int_0^T\varphi(t)\,dt \tag{4.10}$$

Assume moreover that $\varphi(q)>0$ in the state domain. Introduce the equation

$$\left\{\begin{array}{lcl} \dot{\widetilde{q}}_0 & = & \varphi(q) \\ \widetilde{q}_0(0) & = & 0 \end{array}\right.$$

and $\widetilde{q}=(q,\widetilde{q}_0)$ is the enlarged state. Hence (4.9), (4.10) can be written

$$\dot{\widetilde{q}}=\widetilde{X}(\widetilde{q})+u\widetilde{Y}(\widetilde{q}),\ |u|\le 1 \tag{4.11}$$

and let s be the new time parameter defined by

$$ds=\varphi(q(t))\,dt \tag{4.12}$$

and if q' denote the derivative of q with respect to s, (4.9) can be written

$$q'=\overline{X}(q)+u\overline{Y}(q),\ |u|\le 1 \tag{4.13}$$

where $\overline{X}=\psi X$, $\overline{Y}=\psi Y$ and $\psi=\frac{1}{\varphi}$. The optimal control problem becomes a time minimum control problem.

Definition 3. *Consider $\dot{q}=X+uY$. A singular trajectory of the system (X,Y) is a projection of the following equations*

$$\begin{gathered}\dot{q}=\frac{\partial H}{\partial p}\\ \dot{p}=-\frac{\partial H}{\partial q}\\ \langle p,Y\rangle=0\end{gathered} \tag{4.14}$$

where $H=\langle p,X+uY\rangle$, $p\neq 0$. It is called exceptional if $H=0$ and, admissible if $|u|\le 1$ and strictly admissible if $u\in\,]-1,+1[$.

Notation 1. *If X_1 and X_2 are two smooth vector fields, we denote by $[X_1,X_2]$ the Lie bracket computed with the convention*

$$[X_1,X_2](q)=\frac{\partial X_2}{\partial q}(q)\,X_1(q)-\frac{\partial X_1}{\partial q}(q)\,X_2(q)$$

Assumption 5. *Throughout this article we shall assume that* $g = \frac{\mu_e}{r^2}$ *is constant.*

Proposition 2. *In the domain* $\cos\gamma \neq 0$ *there is no exceptional singular arc for the system* (X, Y).

Proof. The singular extremals are located on $\langle p, Y(q)\rangle = 0$. Differentiating twice with respect to t one gets

$$\langle p, [X,Y](q)\rangle = 0$$
$$\langle p, [X,[X,Y]](q)\rangle + u\,\langle p, [Y,[X,Y]](q)\rangle = 0$$

We must compute the Lie brackets Y, $[X,Y]$, $[Y,[X,Y]]$ and $[X,[X,Y]]$ where

$$X = v\sin\gamma\frac{\partial}{\partial r} - \left(g\sin\gamma + k\rho v^2\right)\frac{\partial}{\partial v} + \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right)\frac{\partial}{\partial\gamma}$$
$$Y = \overline{k}\rho v\frac{\partial}{\partial\gamma}$$

and k , $\overline{k}$ are defined by the equations (2.1b) and (2.1c). Since the concept of singular arc is feedback invariant we can replace in our computations X and Y by

$$X = v\sin\gamma\frac{\partial}{\partial r} - \left(g\sin\gamma + k\rho v^2\right)\frac{\partial}{\partial v}$$
$$Y = \frac{\partial}{\partial\gamma}$$

We have then

$$[X,Y] = -v\cos\gamma\frac{\partial}{\partial r} + g\cos\gamma\frac{\partial}{\partial v}$$
$$[Y,[X,Y]] = v\sin\gamma\frac{\partial}{\partial r} - g\sin\gamma\frac{\partial}{\partial v}$$

hence $[X,Y]$ and $[Y,[X,Y]]$ are colinear. Moreover

$$[X,[X,Y]] = k\rho v^2\cos\gamma\frac{\partial}{\partial r} + \left(-kv^3\rho'\cos\gamma + 2k\rho g v\cos\gamma\right)\frac{\partial}{\partial v}$$

The singular extremals are located on Σ': $\langle p, Y\rangle = \langle p, [X,Y]\rangle = 0$ that is $p_\gamma = p_v g - p_r v = 0$. We introduce

$$D = \det\left(Y, [X,Y], [Y,[X,Y]]\right)$$
$$D' = \det\left(Y, [X,Y], [X,[X,Y]]\right)$$
$$D'' = \det\left(Y, [X,Y], X\right)$$

From our previous computations singular arcs are located on $D = D' = 0$ and moreover if they are exceptional they satisfy $D'' = 0$. We have

$$D \equiv 0$$
$$D' = kv^2\cos^2\gamma\left(\rho' v^2 - 3\rho g\right)$$
$$D'' = k\rho v^3\cos\gamma$$

Since $\cos\gamma \neq 0$ the proposition is proved. □

Moreover we have for system (4.9)

Lemma 1. *If* $\cos\gamma \neq 0$ *then*

(1) Y *and* $[X,Y]$ *are independent;*

(2) $[Y,[X,Y]] \in \mathrm{Span}\{Y,[X,Y]\}$

4.4.2. *Analysis of extremals.* Consider the time minimum control problem for system (4.13):

$$q' = \overline{X}(q) + u\overline{Y}(q), \ |u| \leq 1$$

We introduce the following definitions

Definition 4. *The set Σ: $\langle p, \overline{Y}(q)\rangle = 0$ is called the switching surface. Let (q,p,u) be an extremal defined on $[0,T]$; it is called singular if it is contained in Σ, bang if $u=+1$ or $u=-1$ and bang–bang if $u(t)$ is piecewise constant and given a.e. by $u(t) = -\operatorname{sign}\langle p(t), Y(q(t))\rangle$. We denote respectively by γ_+ (resp. γ_-, γ_s) a smooth arc associated to $u=+1$ (resp. $u=-1$, u singular control) and $\gamma_+\gamma_-$ represents an arc γ_+ followed by an arc γ_-.*

Let us calculate Lie brackets. We have

$$\overline{X} = \psi\left(v\sin\gamma\frac{\partial}{\partial r} - \left(g\sin\gamma + k\rho v^2\right)\frac{\partial}{\partial v} + \cos\gamma\left(-\frac{g}{v}+\frac{v}{r}\right)\frac{\partial}{\partial\gamma}\right)$$

$$\overline{Y} = \psi\overline{k}\rho v\frac{\partial}{\partial\gamma}$$

where $\psi = \varphi^{-1}$. Since $\overline{X} = \psi X$, $\overline{Y} = \psi Y$ using for f_1, f_2 smooth functions the formula

$$[f_1X, f_2Y] = f_1f_2[X,Y] + f_1(Xf_2)Y - f_2(Yf_1)X$$

where $Zf = \frac{\partial f}{\partial q}Z(q)$ is the Lie derivative we get

$$\left[\overline{X},\overline{Y}\right] = \psi^2[X,Y] + \psi(X\psi)Y - \psi(Y\psi)X$$

Since $Y = \overline{k}\rho v\frac{\partial}{\partial\gamma}$ and $\psi = f(\rho,v)$ we have $Y\psi = 0$. Hence $\left[\overline{X},\overline{Y}\right] = \psi^2[X,Y] + \psi(X\psi)Y$. Computing $\left[\overline{Y},\left[\overline{X},\overline{Y}\right]\right]$ as before we obtain

Lemma 2. (1) *The set Σ': $\langle p,\overline{Y}\rangle = \langle p,\left[\overline{X},\overline{Y}\right]\rangle = 0$ is given by $\langle p,Y\rangle = \langle p,[X,Y]\rangle = 0$.*
(2) $\left[\overline{Y},\left[\overline{X},\overline{Y}\right]\right] = \psi^3[Y,[X,Y]] \mod \mathrm{Span}\{Y,[X,Y]\}$ *and hence* $\left[\overline{Y},\left[\overline{X},\overline{Y}\right]\right] \in \mathrm{Span}\{Y,[X,Y]\}$.

Moreover,

$$\overline{D} = \det\left(\overline{Y},\left[\overline{X},\overline{Y}\right],\left[\overline{Y},\left[\overline{X},\overline{Y}\right]\right]\right) \equiv 0$$

and

$$\overline{D}'' = \det\left(\overline{Y},\left[\overline{X},\overline{Y}\right],\overline{X}\right) = \frac{k\overline{k}^2\rho\cos\gamma}{C_q^4v^7}$$

and hence $\overline{Y}$, $\left[\overline{X},\overline{Y}\right]$, $\overline{X}$ are a frame in the flight domain where $\cos\gamma \neq 0$. So there exists a, b, c such that

$$\left[\overline{X},\left[\overline{X},\overline{Y}\right]\right] = a\,\overline{X} + b\,\overline{Y} + c\,\left[\overline{X},\overline{Y}\right] \tag{4.15}$$

Long computations give us the crucial result

Lemma 3. *If $\cos\gamma \neq 0$ we have*

(1) $\overline{D}' = \det\left(\overline{Y},\left[\overline{X},\overline{Y}\right],\left[\overline{X},\left[\overline{X},\overline{Y}\right]\right]\right) = -\frac{\beta}{2}\frac{k\overline{k}^3\rho}{C_q^6v^{11}}\cos^2\gamma \neq 0$

(2) $a = -\frac{\beta}{2}\frac{\overline{k}\sqrt{\rho}}{C_q^2v^4}\cos\gamma < 0$

Corollary 1. *If* $\cos\gamma \neq 0$ *there exists no singular trajectory.*

4.4.3. *Application to the classification of extremals near* Σ. Let (q,p,u) be a smooth extremal on $[0,T]$. Differentiating the switching function $\Phi : t \to \langle p(t), \overline{Y}(q(t))\rangle$ we have

$$\dot{\Phi}(t) = \langle p(t), [\overline{X}, \overline{Y}](q(t))\rangle$$
$$\ddot{\Phi}(t) = \langle p(t), [\overline{X}, [\overline{X}, \overline{Y}]](q(t)) + u(t)[\overline{Y}, [\overline{X}, \overline{Y}]](q(t))\rangle$$

We use the results of [10] to classify the extremals near a point $z_0 = (q_0, p_0)$.

(1) Ordinary points. If z_0 belong to $\langle p, \overline{Y}\rangle = 0$, $\langle p, [\overline{X}, \overline{Y}]\rangle \neq 0$, the point z_0 is called of order 1 or ordinary and each extremal curve is locally of the form $\gamma_+\gamma_-$ or $\gamma_-\gamma_+$.

(2) Points of order 2. Let $z_0 \in \Sigma'$: $\langle p, \overline{Y}\rangle = \langle p, [\overline{X}, \overline{Y}]\rangle = 0$. Then if (q,p,u) is a smooth extremal through z_0 the switching function satisfies at z_0:

$$\Phi(t) = \dot{\Phi}(t) = 0$$

and

$$\begin{aligned}\ddot{\Phi}(t) &= \langle p(t), [\overline{X}, [\overline{X}, \overline{Y}]](q(t)) + u(t)[\overline{Y}, [\overline{X}, \overline{Y}]](q(t))\rangle \\ &= \langle p(t), [\overline{X}, [\overline{X}, \overline{Y}]](q(t))\rangle\end{aligned}$$

from lemma 2 which is non zero from lemma 3. Hence both curves corresponding to $u = +1$ and $u = -1$ have a contact of order 2 with respect to Σ and the extremal solutions are represented on figure 3. According to the

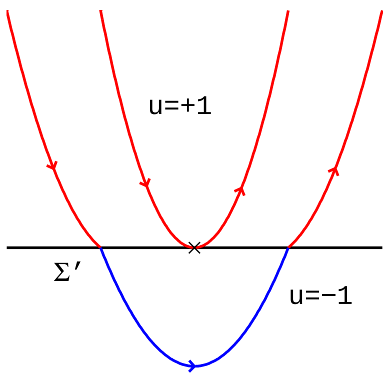

FIGURE 3. extremal solutions $(a > 0)$

classification of [10] the point z_0 is a parabolic point and each extremal is locally bang–bang and of the form $\gamma_+\gamma_-\gamma_+$ or $\gamma_-\gamma_+\gamma_-$.

From this analysis and from the minimum principle we can conclude about small time optimal policy.

Theorem 1. *If $\cos\gamma \neq 0$ each small time optimal policy is of the form $\gamma_-\gamma_+\gamma_-$ where γ_+ is an arc corresponding to $u = \cos\mu = +1$ and γ_- an arc corresponding to $u = -1$ (or $u = 0$ if $\mu \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).*

Proof. According to the time minimum principle, an optimal arc has to satisfy $\widetilde{H} = 0$, $p_0 \geq 0$ where $\widetilde{H} = \left\langle p, \overline{X}(q) + u\overline{Y}(q)\right\rangle + p_0 = 0$. Hence p can be oriented at $z_0 \in \Sigma'$ according to $\left\langle p, \overline{X}(q)\right\rangle \leq 0$. Write

$$\left[\overline{X}, \left[\overline{X}, \overline{Y}\right]\right] = a\,\overline{X} + b\,\overline{Y} + c\,\left[\overline{X}, \overline{Y}\right]$$

and $a < 0$ from lemma 3. Hence from figure 3, only an extremal $\gamma_-\gamma_+\gamma_-$ can be optimal. The assertion is proved. □

4.5. **Geometry of the small time reachable set.** Consider again system in 3–dimension

$$\frac{dq}{ds} = \overline{X}(q) + u\overline{Y}(q)$$

and its time extension in 4–dimension by adding the cost $\frac{ds'}{ds} = 1$. We denote respectively by $R(q_0)$ the small time reachable set and by $\widetilde{R}(q_0, 0)$ the small time reachable set for the extended system. One major research program undertake in [14, 11] using original ideas from Lobry is to evaluate in small dimensions the small time reachable set and its boundary. In particular the following result is basic:

Lemma 4. *Consider system $\left(\overline{X}, \overline{Y}\right)$ in dimension 3 and let $g_1 = \overline{X} + \overline{Y}$ and $g_2 = \overline{X} - \overline{Y}$. Assume g_1, g_2 and $[g_1, g_2]$ linearly independent at q_0 then $R(q_0)$ is bounded by the two surfaces $\gamma_+\gamma_-(q_0)$ and $\gamma_-\gamma_+(q_0)$ and moreover $R(q_0) = \bigcup \gamma_+\gamma_-\gamma_+(q_0)$ (or $\bigcup \gamma_-\gamma_+\gamma_-(q_0)$)*

Actually in theorem 1 we proved more (see also [14]):

Lemma 5. *If $\cos\gamma \neq 0$ the boundary of the small time reachable set for the extended system $\widetilde{R}(q_0, 0)$ is an union of $\widetilde{\gamma}_-\widetilde{\gamma}_+\widetilde{\gamma}_-(q_0, 0)$ and $\widetilde{\gamma}_+\widetilde{\gamma}_-\widetilde{\gamma}_+(q_0, 0)$ where $\widetilde{\gamma}$ denotes the time extended trajectory.*

4.6. **Optimal control of the atmospheric arc.** If we consider the complete set of equations it can be written as a time optimal control problem for a 6–dimension system of the form

$$q' = \overline{X}(q) + u_1\overline{Y}_1(q) + u_2\overline{Y}_2(q)$$

where $u_1 = \cos\mu$ and $u_2 = \sin\mu$. We can use two points of view related to the control device:

- We set $\dot{\mu} = w$ where w is taken as a control bounded by M. The system is then a single input affine control system on the 7–dimensional state space (q, μ).
- We can consider the original system on the 6–dimensional state space. The control $u = (u_1, u_2)$ satisfies $u_1^2 + u_2^2 = 1$. If $\mu \in [0, 2\pi]$, the optimal control problem is equivalent to a sub-Riemannian problem with drift. Indeed if we set $\psi_i = \left\langle p, \overline{Y}_i(q)\right\rangle$, $i = 1, 2$ an extremal normal control is given by $u = \frac{1}{\|\psi\|}(\psi_1, \psi_2)$.

We must analyze the existence of abnormal extremals and the number of oscillations and switchings of optimal trajectories.

4.7. **Conclusion about this section.** Using minimum principle, Lie brackets and geometric methods we have evaluated the small time reachable set and solved the small time optimal control problem for system I. In particular we have obtained bounds on the number of switchings. The main property of system I is that $\left[\overline{Y},\left[\overline{X},\overline{Y}\right]\right]$ belong to Span $\left\{\overline{Y},\left[\overline{X},\overline{Y}\right]\right\}$. This is connected to the feedback linearizability if system I. For the global aspect one needs to analyze the global proportion of the switchings function Φ using convexity analysis and Rolle theorem. Our study is a preliminary step in order to evaluate the reachable set for the full system of equation without the state constraints and nearby the state constraints. We shall analyze the structure of the reachable set for system nearby the constraints in the next section.

5. Optimal control with state constraints

In this section we analyze the optimal control problem for system I, taking into account the constraints. We recall a minimum principle from [12] adapted to our situation. Our contribution is to make a direct evaluation of the small time reachable set for the constrained system using the previous computations of section 4 and a normal form.

When dealing with constrained systems the main concept is the concept of order of the constraint that we define next before to state the minimum principle adapted to our analysis.

5.1. **A minimum principle.** We consider the single input affine control system

$$\dot{q} = f(q) + ug(q) \quad |u| \leq 1$$

and a cost to be minimized of the form

$$J(u) = G(x(T))$$

where the transfer time T is fixed and q is constrained to

$$c(q) \leq 0$$

The boundary conditions are

$$\begin{aligned} q(0) &= q_0 \\ \Phi(x(T)) &= 0 \end{aligned}$$

The problem is denoted by $(\mathcal{P}_0)$ and can be imbedded into the one parameter family of problems $(\mathcal{P}_\alpha)$ where the constraints set is taken as

$$c(q) \leq \alpha,\ \alpha \text{ small}$$

The important concept is the concept of order of the constraint.

Definition 5. *The absolute (or generic) order of the constraint is the first integer $n+1$ such that*

$$\begin{aligned} & g\left(f^0 c\right) \equiv g\left(f^1 c\right) \equiv \cdots \equiv g\left(f^{n-1} c\right) \equiv 0 \\ & g\left(f^n c\right) \neq 0 \end{aligned}$$

where the vector fields f, g acts on c by Lie derivative.

Definition 6. *A boundary arc $t \mapsto \gamma_b(t)$ is a solution of the system contained in $c = 0$. If the constraint is of order n it can be generically computed by differentiating n times the constraint and solving the linear equation*

$$c^{(n)} = f^n c + ug\left(f^{n-1}c\right) = 0 \tag{5.1}$$

A boundary arc is contained in

$$c = \dot{c} = \cdots = c^{(n-1)} = 0 \tag{5.2}$$

and the constraints $\dot{c} = \cdots = c^{(n-1)} = 0$ are called the secondary constraints.

We denote by u_b the feedback control $\frac{-f^n c}{g(f^{n-1}c)}$ which allows to remain in the constrains set.

Let's now formulate the Maurer minimum principle [12]:

Assumption 6 (General assumption)**.** *We assume that the following conditions hold on a boundary arc $s \mapsto \gamma_b(s)$, $s \in [0, t]$:*

- **(H_6):** $g\left(f^{n-1}c\right)|_{\gamma_b} \neq 0$ *(n being the order)*
- **(H_7):** $|u_b(t)| < 1$ *i.e. the boundary feedback control is admissible and not saturating.*

Necessary conditions. Define the Hamiltonian by

$$H(q, u, p, \eta) = \langle p, f + ug \rangle + \eta c \tag{5.3}$$

where η is a Lagrange multiplier of the constraint set. The necessary conditions of the minimum principle are the following:

Condition 1.

(1) *There exists $\eta(t) \geq 0$, a real number $\eta_0 \geq 0$ and δ such that the adjoint (row) vector satisfies*

$$\dot{p} = -p\left(\frac{\partial f}{\partial q} + u\frac{\partial g}{\partial q}\right) - \eta\frac{\partial c}{\partial q} \tag{5.4}$$

$$p(T) = \eta_0 \frac{\partial \Phi}{\partial q}(q(T)) + \delta\frac{\partial G}{\partial q}(x(T)) \tag{5.5}$$

(2) *The function $\eta(t)$ satisfies $\eta(t)\, c(q(t)) = 0$ $\forall t \in [0, T]$ and is continuous on the interior of the boundary arc.*

(3) *The jump condition at a contact point or a junction time t_1 is*

$$p\left(t_1^+\right) = p\left(t_1^-\right) - \nu_1 \frac{\partial c}{\partial q}(q(t_1)),\ \nu_1 \geq 0 \tag{5.6}$$

(4) *The optimal control $u(t)$ minimizes the Hamiltonian, i.e.*

$$H(q(t), u(t), p(t), \eta(t)) = \min_{|u| \leq 1} H(q(t), u, p(t), \eta(t)) \tag{5.7}$$

Remark 1. *In this minimum principle, only the constraint c is penalized in H; others choices are possible using the secondary constraints, see* [8, 13]*.*

Remark 2. *There exist a general minimum principle without assumption (*H_6*), see for instance* [9] *where the adjoint equation (5.4) takes the form*

$$p(t) = -\int p(s)\left(\frac{\partial f}{\partial q} + u\frac{\partial g}{\partial q}\right) ds - \int \frac{\partial c}{\partial q} d\mu_i$$

where μ_i is a measure supported on the set $c = 0$. Our principle is much more precise because from (5.4) the measure is of the form

$$d\mu_i = \eta(t)\, dt$$

where η is $\mathcal{C}^0$. This additional regularity comes from assumption (H_6) *and at non generic point where $g\left(f^{n-1}c\right)$ vanishes η can blow up.*

The case where T is not fixed can be deduced from the case where T is fixed. We introduce a new variable $z = T$ and the system

$$\begin{aligned}\frac{dt}{ds} &= z\\ \frac{dq}{ds} &= \left(f(q) + ug(q)\right) z\\ \frac{dz}{ds} &= 0\end{aligned}$$

We have $s = \frac{t}{T}$ and the trajectories are parametrized by $s \in [0,1]$. The new transfer time is 1.

An important research program is to analyze the solutions of the minimum principle with constraints. This analysis is outlined in [12]. An interesting point of view is to analyze the open loop solution deduced from the problem without constraints by analyzing the bifurcation of an unconstrained optimal solution when the constraint $c(q) \leq \alpha$ becomes active.

Next we adopt a different approach based on the evaluation of the small time reachable set near the constraints. It will provide necessary and sufficient optimality conditions.

5.2. **A direct approach.**

5.2.1. *Order of the constraint.* Consider in the shuttle problem the constraint on thermal flux

$$c_1 = C_q\sqrt{\rho}v^3 \leq \alpha,\ \rho = \rho_0 \mathrm{e}^{-\beta r}$$

and $\dot{c}_1 = \varphi_1(r,v) + \varphi_2(r,v)\sin\gamma = 0$ is a secondary constraint. Moreover $\ddot{c}_1 = \varphi_3(r,v,\gamma) + u\varphi_4(r,v)\cos\gamma$ where $\varphi_4(r,v) = -\overline{k}C_q\rho^{\frac{3}{2}}\left(3gv^3 + \frac{\beta}{2}v^5\right) \neq 0$.

Similarly for the normal acceleration $c_2 = \gamma_{n_0}\rho v^2$ we get

$$\dot{c}_2 = -\gamma_{n_0}\left(2k\rho^2v^3 + \left(\beta\rho v^3 + 2g\rho v\right)\sin\gamma\right)$$

i.e. $\dot{c}_2 = \varphi_5(r,v) + \varphi_6(r,v)\sin\gamma = 0$, $\varphi_6(r,v) = -\gamma_{n_0}\rho\left(\beta v^3 + 2gv\right)$ is a secondary constraint and $\ddot{c}_2 = \varphi_7(r,v,\gamma) + u\varphi_8(r,v)\cos\gamma$ with $\varphi_8(r,v) = -\overline{k}\gamma_{n_0}\rho^2\left(\beta v^4 + 2gv^2\right) \neq 0$. Hence we prove:

Lemma 6. *For the space shuttle if $\cos\gamma \neq 0$ the constraints on the thermal flux and on the normal acceleration are of order 2 and* (H_6) *is satisfied along a boundary arc.*

5.2.2. *Evaluation of the small time reachable set for the constrained system.* It is based on the following normal form. Consider system $\dot{q} = X + uY$, $|u| \leq 1$, $q = (x,y,z)$ and the constraint $c(q) \leq \alpha$, $\alpha \simeq 0$. We compute a normal form in the geometric configuration of the shuttle system near $q_0 \in c(q) = 0$.

- **Normalization 1.** We let $q_0 = 0$. Assume $Y(0) \neq 0$. Hence Y can be identified locally to $\frac{\partial}{\partial z}$. Diffeomorphisms preserving Y are $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ where $\frac{\partial \Phi_1}{\partial z} = \frac{\partial \Phi_2}{\partial z} = 0$ and $\frac{\partial \Phi_3}{\partial z} = 1$. In our problem the constraint is of order 2, hence $Yc = 0$ near 0 and Y is tangent to all the surfaces $c = \alpha$. Hence $\frac{\partial c}{\partial z} = 0$.
- **Normalization 2.** Since c is not depending upon z using a diffeomorphism preserving $Y \sim \frac{\partial}{\partial z}$ we can normalize c to $c(x,y) = x$. The system can be written
$$\begin{aligned} \dot{x} &= X_1(q) \\ \dot{y} &= X_2(q) \\ \dot{z} &= X_3(q) + u \end{aligned}$$
and $c = x$. The secondary constraint is $\dot{x} = 0$ and we assume that $x = \dot{x} = 0$ is an arc σ passing through $q_0 = 0$. If we keep the affine approximation sufficient for our analysis we obtain a system which can be written
$$\begin{aligned} \dot{x} &= a_1 x + a_2 y + a_3 z \\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 z \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + u \end{aligned}$$
where σ is approximated by the straight line $x = a_2 y + a_3 z$. If $b_0 \neq 0$ (generic case) we can assume $b_0 = 1$.
- **Normalization 3.** Changing z into $-z$ and u into $-u$ if necessary and using a transformation of the form $Z = \alpha y + z$ one can identify σ to $x = z = 0$ and the system can be written
$$\begin{aligned} \dot{x} &= a_1 x + a_3 z \\ \dot{y} &= 1 + b_1 x + b_2 y + b_3 z \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + u \end{aligned} \tag{5.8}$$
where $a_3 > 0$. If moreover the boundary arc is admissible and not saturating (assumption (H$_7$)) we have the condition $|c_0| < 1$.

Theorem 2. *Consider the problem of time minimization in $\dot{q} = \overline{X}(q) + u\overline{Y}(q)$, $q \in \mathbb{R}^3$ subject to $c(q) \leq 0$. Let $q_0 \in \{c = 0\}$ and assume the following:*

(1) *Near q_0, $\left[\overline{Y}, \left[\overline{X}, \overline{Y}\right]\right] \in \operatorname{Span}\left\{\overline{Y}, \left[\overline{X}, \overline{Y}\right]\right\}$*

(2) *$\overline{X}, \overline{Y}, \left[\overline{X}, \overline{Y}\right]$ are linearly independent at q_0 and*
$$\left[\overline{X}, \left[\overline{X}, \overline{Y}\right]\right](q_0) = a\overline{X}(q_0) + b\overline{Y}(q_0) + c\left[\overline{X}, \overline{Y}\right](q_0)$$
with $a < 0$

(3) *The constraint $c = 0$ is of order 2 and assumption (H$_6$) and (H$_7$) are satisfied at q_0*

then the boundary arc through q_0 is small–time optimal if and only if $\gamma_-(q_0)$ is contained in the domain $c \geq 0$.

Proof. From lemma 4, we know that each small time reachable point from q_0 can be reached by an arc $\gamma_+\gamma_-\gamma_+$ and $\gamma_-\gamma_+\gamma_-$ and from theorem 1 we know that the small time optimal arc is of the form $\gamma_-\gamma_+\gamma_-$ for the unconstrained system.

Let the constrained system written as (5.8) in the normal coordinates where $q_0 = 0$ and the boundary arc $\gamma_b(t)$ is identified to $(0, t, 0)$. Let $B = \gamma_b(t)$, $t > 0$

small enough. Let $u = +1$ or $u = -1$. For a trajectory with $q(0) = 0$ we have the following approximations:

$$z(t) = (c_0 + u)\, t + o(t)$$
$$x(t) = a_3 (c_0 + u) \frac{t^2}{2} + o(t)$$

Hence the projections of the arcs $\gamma_+\gamma_-\gamma_+$ and $\gamma_-\gamma_+\gamma_-$ joining 0 to B in the plane (x, z) are loops denoted $\widetilde{\gamma}_+\widetilde{\gamma}_-\widetilde{\gamma}_+$ and $\widetilde{\gamma}_-\widetilde{\gamma}_+\widetilde{\gamma}_-$ and are represented on figure 4.

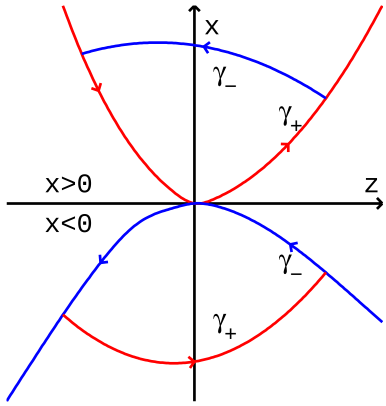

FIGURE 4. Projection of the arcs $\gamma_+\gamma_-\gamma_+$ and $\gamma_-\gamma_+\gamma_-$

In particular, we proved the following.

Lemma 7. *The loops $\widetilde{\gamma}_-\widetilde{\gamma}_+\widetilde{\gamma}_-$ (resp. $\widetilde{\gamma}_+\widetilde{\gamma}_-\widetilde{\gamma}_+$) are contained in the domain $x < 0$ (resp. $x > 0$).*

We can now end the proof of the theorem (the assertions concern system $\left(\overline{X}, \overline{Y}\right)$).

If the arc $\gamma_-\gamma_+\gamma_-$ to join 0 to B is contained in the domain $c \leq 0$ it is time minimal and the boundary arc is not optimal. If the arc $\gamma_-\gamma_+\gamma_-$ is contained in $c \geq 0$ then we can join 0 to B by an arc $\gamma_+\gamma_-\gamma_+$ in $c \leq 0$. But the analysis of section 4 replacing $\min t$ by $\max t$ shows that such an arc is time maximal. Hence a bang–bang arc $\gamma_+\gamma_-\gamma_+$ in the domain $c \leq 0$ joining 0 to B cannot be optimal. Then the boundary arc γ_b is optimal. □

Moreover

Corollary 2. *If a boundary arc γ_b is small time optimal then there exist optimal trajectories of the form $\gamma_-\gamma_+\gamma_b\gamma_+\gamma_-$.*

5.2.3. *Application to the shuttle.* For the shuttle we have $a < 0$, so we have to consider loops $\gamma_-\gamma_+\gamma_-$ where γ_- corresponds to $\cos\mu = 0$ or $\cos\mu = -1$. From the computations of section 5.2.1 we have for both constraints c_1, c_2:

$$\ddot{c}_i = \Phi(r, v, \gamma) + u \cos\gamma \overline{\Phi}(r, v)$$

where $\overline{\Phi} < 0$. Hence $\ddot{c}_i$ is minimal when $\mu = 0\,^\circ$ i.e. $u = +1$. Assume that the parameters of the problem are such that assumption (H_7) is satisfied. Then the arcs γ_- through the boundary points are contained in the non admissible domain and the boundary arc is optimal.

6. Conclusion

We have outlined the geometric research program to analyze the optimal control of the atmospheric arc for the space shuttle. Our tools are necessary optimality conditions and evaluation of the small time reachable set. Near the constraints the evaluation is related to the classification of pairs of vector fields near a surface. This problem is common to several problems met in optimal control: classification of extremals near the switching surface, optimal control with targets and so on...

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