Theoretical and experimental study of two discrete coupled Nagumo chains

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Abstract

We analyze front waves (kink and anti-kink) propagation and pattern formation in a system composed of two coupled discrete Nagumo chains using analytical and numerical methods. In the case of homogeneous interaction among the chains, we show the possibility of the effective control on wave propagation. In addition, physical experiments on electrical chains confirm all theoretical behaviors.
I. INTRODUCTION

In a variety of spatially extended systems arising in many areas of science, the competition between stationary states may yield the appearance of propagating front waves or kinks. These processes, defining the interfaces between different states of the medium, occur for example, in some problems of cardiology, neurophysiology, chemistry and physics (see e.g. [1–3]). In particular, in a wide class of systems possessing front wave solutions, a significant place is taken by reaction-diffusion systems (RD-systems). At present, among the RD-systems, the most deeply studied is the one-component Nagumo equation which corresponds to the well-known FitzHugh-Nagumo system without the recovery variable. There are two cases of Nagumo equation considered in the literature, with respectively continuous and discrete spatial coordinate.

In the first case, this equation takes the form

\[ u_t = du_{xx} + F(u), \quad (1.1) \]

where \( u(x, t) \) is the state variable and \( d \) is the diffusion constant. The nonlinearity, providing the bistability of the medium, is expressed by \( F(u) = -(u - m_1)(u - m_2)(u - m_3) \) with \( 0 < m_1 < m_2 < m_3 \). It is well-known (see, e.g., [4,5]) that equation (1.1) has front wave or kink solutions of the form \( u(x, t) = U(x - ct) \equiv U(\xi) \) with

\[ U(\xi) \equiv \frac{m_3 + m_1 \exp \left( \pm \frac{m_3 - m_1}{\sqrt{2d}} \xi \right)}{1 + \exp \left( \frac{m_3 - m_1}{\sqrt{2d}} \xi \right)}, \quad (1.2) \]

and

\[ c = \pm \sqrt{d/2}(m_1 + m_3 - 2m_2). \quad (1.3) \]

The signs \( \pm \) correspond to kink and anti-kink solution, respectively.

The discrete Nagumo equation has the form

\[ \dot{u}_j = d(u_{j-1} - 2u_j + u_{j+1}) + F(u_j), \quad (1.4) \]
with the dot accounting for the time derivative and \( j \) defining a space lattice point \((j \in \mathbb{Z})\) or discrete space coordinate, \( d \) being the coupling coefficient. Equation (1.4) is more preferable when the activity of the medium is provided by localized units in the junctions of space lattice. Take, for instance, a myelinated nerve fiber [2] when the membrane activity is localized mostly in Ranvier nodes coupled by myelinated (passive) parts of the axon. Another example is the heart tissue composed of a number of interacting cardiac cells suitably distributed in space and coupled with gap junctions which may be approximated with nearest neighbors diffusive coupling [6,7]. From an engineering point of view, discrete RD-systems based on nonlinear electric RD-lattices [8–11] or biological enzyme transistor circuits [12] can be used for various information processing problems.

Although Eq. (1.4) presents qualitatively similar behaviors to Eq. (1.1), it also displays some different properties [13–22]. In particular, there exists a critical value of the coupling coefficient \( d = d^*(m_1, m_2, m_3) \) above or under which front propagation is possible or not.

(i) For \( d > d^* \), front wave solutions of two types (kinks and anti-kinks) are possible in equation (1.4) and do not differ qualitatively from solutions of (1.1).

(ii) For \( d \leq d^* \), the propagation of front waves is impossible in equation (1.4) for any relations between parameters \( m_1, m_2 \) and \( m_3 \). This so called propagation failure phenomenon does not exist in the continuous case modeled by equation (1.1).

Figure 1 illustrates the dependence of front waves speed in (1.4) on parameter \( m_2 \), for a given \( d \). By contrast with the continuous case depicted by (1.1) and shown by dashed curves in Fig. 1, there is an interval of \( m_2 \) in which \( c = 0 \), corresponding to the propagation failure. However, in many cases, one has to consider not only a single chain, but systems consisting in different interacting chains. For example, sciatic nerve of animals consists of several hundred fibers, with some spatially distributed electric contacts. In this context, there is a growing interest devoted to the understanding of interfiber interactions [23–26].

The goal of this paper is to investigate the dynamics of a system composed of two coupled discrete chains modeling two coupled FitzHugh-Nagumo chains without recovery variables.

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Then, our system is expressed under the following normalized form:

\[
\begin{aligned}
\dot{u}_j &= f(u_j) + d(u_{j-1} - 2u_j + u_{j+1}) - h_j(u_j - v_j), \\
\dot{v}_j &= f(v_j) + d(v_{j-1} - 2v_j + v_{j+1}) - h_j(v_j - u_j),
\end{aligned}
\]  

(1.5)

where the cubic function \(f(w)\) is given by

\[f(w) = w(w - 1)(a - w), \quad 0 < a < 1,\]

\(h_j\) and \(d\) being respectively the interchain and intrachain coupling coefficients (\(h_j\) is a vector with \(j\) defining a space chain point \((j = 1, 2, \ldots, N)\)). However, in a sake of simplicity, we will restrict our study to the homogeneous interchain coupling case, that is, \(h_j = h, \forall j\). Furthermore, we impose Neumann boundary conditions to the system (1.5):

\[w_0 = w_1, \quad w_{N+1} = w_N.\]  

(1.6)

The paper is organized as follows. In Sect. II, we discuss the main properties of the system (1.5) from its phase space point of view. In Sect. III, we inquire into the existence of patterns, high multistability and wave propagation failure in system (1.5). Then, in Sect. IV, we study the dynamics of the system resulting in wave motions. Finally, experimental results on a real system composed of two coupled bistable electrical chains are presented in Sec. V.

II. MAIN PROPERTIES OF SYSTEM (1.5)

A. The confinement of trajectories

Let us show in this section that all trajectories of system (1.5) are confined. In this purpose, we introduce the \(r\)-family of regions in the phase space

\[\Omega_r = \{u, v : -r \leq u_j \leq 1 + r, \quad -r \leq v_k \leq 1 + r, \forall j, k = 1, 2, \ldots, N\},\]
with \( r \geq 0 \) being an arbitrary parameter. Let us consider the vector field of (1.5) at the boundaries of each of such regions. It follows from (1.5)

\[
\dot{u}_j \mid_{\{u_j=-r, -r \leq u_i \leq 1+r, i \neq j\}} = f(-r) + d(u_{j-1} + 2r + u_{j+1}) + hr + hv_j \geq f(-r) > 0
\]

\[
\dot{u}_j \mid_{\{u_j=1+r, -r \leq u_i \leq 1+r, i \neq j\}} = f(1 + r) + d(u_{j-1} - 2(1 + r) + u_{j+1}) - h(1 + r) + hv_j \leq f(1 + r) < 0.
\]

Similarly, we determine the orientation of \( v_j \)-components on the boundary of \( \Omega_r \). It coincides with the orientation of \( u_j \)-components. Therefore, for \( r > 0 \), the vector field of the system (1.5) at the boundary of each \( \Omega_r \)-region is oriented inwards. Hence, all trajectories of the system (1.5) with initial conditions outside \( \Omega_0 \) come into this region as time proceeds and do not leave \( \Omega_0 \). Then, in the following, we consider the dynamics of (1.5) in \( \Omega_0 \). Note that the vector field at the boundary of \( \Omega_0 \) is everywhere oriented inwards excluding the two “angle” points, \( O_0(u_j = v_j = 0) \), \( O_1(u_j = v_j = 1) \), \( j = 1, 2, \ldots, N \), which are the steady states of the system (1.5). Then,

\[
0 \leq u_j \leq 1, \ 0 \leq v_k \leq 1, \text{ for } t > 0, \ \forall j, k = 1, 2, \ldots, N. \tag{2.1}
\]

B. The gradient property of the system

Let us consider the function

\[
U = \sum_{j=1}^{N} \left[ \frac{d}{2}(u_{j+1} - u_j)^2 + \frac{d}{2}(v_{j+1} - v_j)^2 - \int_{0}^{u_j} f(\eta) d\eta + \int_{0}^{v_j} f(\eta) d\eta + \frac{h}{2}(u_j - v_j)^2 \right]. \tag{2.2}
\]

Using \( U \), the system (1.5) can be rewritten in the following form:

\[
\dot{u}_j = - \frac{\partial U}{\partial u_j}, \ \dot{v}_j = - \frac{\partial U}{\partial v_j}.
\]
which shows that (1.5) is a gradient system. Hence, the attractors in the $R^N$ phase space can only be steady states of (1.5) [27]. Then, any initial condition tends to one of the stable steady states corresponding to a local minimum of function $U$.

III. SPATIAL PATTERNS

In the “physical” space $\{(Z,R)\}$, each of the stable steady states defines a Turing-like pattern with the spatial profile corresponding to the distribution of the steady state coordinates. For example, steady states $O_0$ and $O_1$ correspond to the stable homogeneous states of the medium. Let us find the maximum number of possible stable steady states, hence corresponding to steady patterns. For this purpose, we use the invariant domains technique (see [28] for details).

For convenience, we define the vector $w = (u_1, u_2, \ldots, u_N, v_1, v_2, \ldots, v_N)^T$, the superscript $T$ denoting the transpose operator, and the following regions in the phase space

$$
\Omega_0^i = \{w: 0 \leq w_i \leq q, 0 \leq w_k \leq 1, \forall k \neq i\}
$$

$$
\Omega_1^i = \{w: 1 - p \leq w_i \leq 1, 0 \leq w_k \leq 1, \forall k \neq i\},
$$

with $0 < q, p < 1$. Let us show that, for parameter values taken from the region

$$
D_{ch} = \begin{cases} 
2d + h < \frac{a^2}{4}, & \text{if } a < \frac{1}{2}, \\
2d + h < \frac{(1 - a)^2}{4}, & \text{if } a \geq \frac{1}{2},
\end{cases}
$$

there exist $p$ and $q$ ensuring that the vector field of system (1.5) at the boundaries of each of these regions, $\Omega_0^i$ and $\Omega_1^i$, is oriented inward them. For example, we consider the behavior of $v_j$-components at the boundaries of $\Omega_0^i$ and $\Omega_1^i$. Taking into account (2.1), we find

$$
\dot{v}_j \mid_{(v_j = q, 0 \leq w_k \leq 1, k \neq j)} = d(v_{j-1} - 2q + v_{j+1}) + f(q) - hq + hu_j \\
\leq d(-2q + 2) + f(q) - hq + h < 0.
$$
Let us demand the negativeness of the derivative (3.2). It is satisfied for any values of \( q \) obeying to

\[
\frac{a}{2} - \sqrt{\frac{a^2}{4} - 2d - h} < q < \frac{a}{2} + \sqrt{\frac{a^2}{4} - 2d - h}. \tag{3.3}
\]

Similarly, at the boundary of region \( \Omega^1_1 \), we find

\[
\dot{v}_j |_{\{v_j=1-p, 0\leq w_k \leq 1, k\neq j\}} = d(v_{j-1} - 2(1 - p) + v_{j+1}) + f(1 - p) - h(1 - p) + hu_j \geq -2d(1 - p) + f(1 - p) - h(1 - p) > 0. \tag{3.4}
\]

The values of parameter \( p \) ensuring the positiveness of the derivative (3.4) are defined by

\[
\frac{1-a}{2} - \sqrt{\frac{(1-a)^2}{4} - 2d - h} < p < \frac{1-a}{2} + \sqrt{\frac{(1-a)^2}{4} - 2d - h}. \tag{3.5}
\]

In the same manner, we determine the orientation of components \( u_j \) at boundary planes \( \{u_j = q\} \) and \( \{u_j = 1-p\} \) of regions \( \Omega^0_j \) and \( \Omega^1_j \). By construction, the boundary of each of the regions \( \Omega^0_j \) consists of two parts, one of them being formed by planes \( \{w_j = q\} \) and the other one by the boundary of region \( \Omega_0 \) (see Sect. II). Therefore, at the boundary of each of regions \( \Omega^0_j \), the vector field of the system (1.5) is oriented inwards. Similarly, we obtain that trajectories of (1.5) intersect the boundaries of regions \( \Omega^1_j \) inwards. Let us fix an arbitrary sequence of length \( N \) composed of two symbols \( a_j \in \{0;1\} \), and consider the intersection \( J = \cap \Omega^2_j \) (see Fig. 2(a)). \( J \) being represented by the direct product of the segments of the coordinate axes, it is a convex compact set. Besides, the boundary of \( J \) is formed by the boundaries of regions \( \Omega^0_j \) and \( \Omega^1_j \), and hence, the trajectories of system (1.5) intersect this boundary inward the region \( J \). Obviously, the set \( J \) contains at least one attractor of system (1.5). Using the gradient property of the system, we find that this attractor can be represented only by a steady state. Since there are \( 2^N \) sets or \( J \)-type regions, there exist \( 2^N \) stable steady states in the phase space of system (1.5).

Thus, system (1.5) displays high multistability. Since the steady states can be encoded by arbitrary sequences of two symbols, the possible pattern profiles in \( \{Z, R\} \) are extremely diverse, varying from regular to complex disordered configurations. For illustration, we may
obtain a disordered pattern on \( u \)-components and a regular one on \( v \)-components, as shown in Fig. 3. Note that the existence of a wealth of steady patterns does not allow any wave-like motion in system (1.5). In fact, the origin of wave propagation failure, quite typical in discrete bistable systems, lies in the existence and stability of patterns: the regions \( \Omega^i_j \) \((i \in \{0, 1\}) \) estimate the steady state attraction basins. Their initial conditions belonging to one of these regions, kinks or anti-kinks are attracted by the corresponding steady state. Thus, the parameter values taken in region \( D_{ch} \) give sufficient conditions to observe this phenomenon.

IV. TRAVELING WAVES

For illustration, in this section, we treat the system (1.5) as two coupled chains divided in the real space. First, we note that results of paper [29] ensure the complete interchain synchronization of all motions for strong enough interchain coupling

\[
h_j > \frac{a^2 - a + 1}{6}.
\] (4.1)

Then, in the \( R^N \) phase space of the system (1.5), there is a \( N \)-dimensional (synchronization) manifold which attracts all motions. In this manifold, motions are governed by a system of type (1.4) with \( F(u) = f(u) \). In this case, traveling waves dynamics does not depend on the coefficients \( h \) and is illustrated by Fig. 1. We will now consider the dynamics of the system for small enough interchain interaction.

A. Front waves, slowing down, stopping and about-turn

Let us consider a front wave or kink propagating in the first chain (\( u \)-components), while the second chain (\( v \)-components) is in the vicinity of the homogeneous steady state \( O_0 \). For convenience, we call this state “unexcited”, because all elements have the coordinates close to zero. We switch on the interaction and analyze how small but non-zero coupling acts on the propagating front.
Let us introduce the following $\gamma$-family of regions in the phase space:

$$V^0(\gamma) = \{u, v : 0 \leq v_j \leq \gamma, 0 \leq u_k \leq 1, \forall j, k = 1, 2, \ldots, N\}.$$  

Then, there exist such values of $\gamma$ for which the trajectories of system (1.5) intersect the boundary of $V^0(\gamma)$ inward this region. Indeed, it is satisfied at the parts of $V^0(\gamma)$ inherited from region $\Omega_0$ (see Sect. II and Fig. 2 (b)). Considering the rest part of $V^0(\gamma)$ and using (2.1), we obtain from (1.5)

$$\dot{v}_j \mid_{(v_j=\gamma, 0 \leq u_k \leq \gamma, k \neq j, 0 \leq u_j \leq 1)} = f(\gamma) + d(v_{j-1} - 2\gamma + v_{j+1})$$

$$-h \gamma + hu_j \leq f(\gamma) - h \gamma + h.$$  

The negativeness of the derivative (4.2) is fulfilled for

$$\gamma_0 \equiv \frac{a}{2} - \sqrt{\frac{a^2}{4} - h} < \gamma < \frac{a}{2} + \sqrt{\frac{a^2}{4} - h}.$$  

Hence, for (4.3) the vector field of (1.5) is oriented inward these regions. Then, if the initial conditions in system (1.5) are taken inside $V^0(\gamma_0)$, the corresponding trajectory satisfies

$$0 \leq v_j \leq \gamma_0, 0 \leq u_k \leq 1, \text{ for } t > 0, \forall j, k = 1, 2, \ldots, N.$$  

Since $\gamma_0 < a/2$, conditions (4.4) mean that components $v_j$, for any $t > 0$, stay below the “excitation threshold” (the definition of “unexcited” chain) and components $u_j$ may take arbitrary values within the region $\Omega_0$.

Introducing a fixed positive parameter $\varepsilon$ with an infinitely small value, it is easy to show that for

$$h \leq a\varepsilon + O(\varepsilon^2),$$  

the parameter $\gamma_0$ satisfies the inequality $\gamma_0 \leq \varepsilon$. In this case, motions in the second chain are of order of an infinitely small value and, hence, the chain evolves in the vicinity of its steady state $O_0$. In spite of the smallness of $h$, expressed in (4.5), the dynamics of the first chain becomes quite different. Indeed, in this case,

$$h(u_j - v_j) = hu_j + O(\varepsilon^2).$$  

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Then, in the first order approximation on $\varepsilon$, the dynamics of the first chain is defined by the system (1.4) with $F(u) = f(u) - hu$. The zeros of $F(u)$ are given by

$$m_1 = 0, \quad m_{2,3} = \frac{1 + a}{2} \mp \sqrt{\frac{(1 - a)^2}{4} - h}. \quad (4.7)$$

Thus, in the case of weak interchain coupling, the second chain stays in its “unexcited” state, while the first chain supports the front propagation with a new velocity. Indeed, according to (4.7), we obtain

$$m_2(h) > m_2(0) = a. \quad (4.8)$$

Since the kink velocity is a monotonically decreasing function of parameter $m_2$ (Fig. 1), the influence of the second chain, $h > 0$, tends to decrease the velocity of the kink in chain 1. Analyzing the diagram in Fig. 1 leads to the following possibilities concerning the dynamics of the kink:

(a) the kink is propagating with a smaller velocity, $c(h) < c(0)$;

(b) the kink stops to propagate, $c(h) = 0$;

(c) the kink reverses and propagates backward, $c(h) < 0$.

Such behaviors of the kink have been verified in numerical simulations of system (1.5). For example, the velocity-coupling dependence $c = c(h)$, obtained for $a = 0.45$ and shown in Fig. 4, involves the three possibilities successively. For comparison, corresponding curves obtained with the first-order perturbation analysis (4.5)–(4.7) are shown by the dashed lines. There is a good qualitative (and quantitative for $h \to 0$) agreement with direct simulation of system (1.5). The case (a), realized for smallest values of $h$, is illustrated in Fig. 5. The front propagating in the first chain for $h = 0$ with the definite velocity becomes slower when switching on the interaction at $t = t_0$. This corresponds to the different angle shown in the space-time plot, $(j, t)$, in Fig. 5. The second chain stays “unexcited”. The front stops with increasing $h$. The stopping front results in a “kink-like” steady pattern, hence...
in the propagation failure. By contrast to the description of section III, the origin of this phenomenon is not caused by internal dynamics for small $d$, but by the interchain dynamics with non-vanishing $h$. Further increase of $h$ leads to the kink reversing as shown in Fig. 6. Endly, the break of the velocity curves in Fig. 4 corresponds to interchain synchronization, that is, the second chain displays a kink identical to the original. As mentioned earlier, the dynamics in this case is defined by the equations for a single chain (1.4), hence the velocity of both synchronized kinks is equal to the unperturbed case $h = 0$.

Note that the increase of intrachain coupling $d$ tends to decrease the domain of zero velocities (Fig. 4). In the limit case of a continuous medium, described by PDE (1.1), that is for $d \gg 1$, we may expect only one point with $c = 0$.

B. Front waves speeding up

Let us consider now a front wave or kink propagating in the first chain ($u$-components), while the second chain ($v$-components) is in the vicinity of the homogeneous steady state $O_1$. We refer to this state as “excited” one, because all elements have their coordinate close to 1. Considering the weak homogeneous interaction ($h > 0$) between the two chains with such initial conditions, and processing as for the previous case, one can show that, for $h > \frac{(1 - a)^2}{4}$, there is a region in the $R^{2N}$-phase space which cannot be left by the trajectories of system (1.5). It has the form (see Fig. 2(b))

$$V^1(\beta_0) = \{u, v : \beta_0 \leq v_j \leq 1, 0 \leq u_k \leq 1, \forall j, k = 1, 2, \ldots, N\},$$

with

$$\beta_0 = \frac{1 + a}{2} + \sqrt{\frac{(1 - a)^2}{4} - h}.$$ 

Thus, for the trajectories with initial conditions taken in region $V^1(\beta_0)$, we obtain the restriction

$$\beta_0 \leq v_j \leq 1, 0 \leq u_k \leq 1, \text{ for } t > 0, \forall j, k = 1, 2, \ldots, N.$$  \hspace{1cm} (4.9)
Since $\beta_0 > a$, such a state of the second chain can be considered as “excited”, in the sense that all $v_j$-components have large enough values (in the vicinity of steady state $O_1$).

Let us fix an arbitrary infinitely small $\varepsilon$ such that

$$1 - \beta_0 < \varepsilon,$$

and

$$h \leq (1 - a)\varepsilon + O(\varepsilon^2). \quad (4.10)$$

From (4.10) we obtain

$$h(v_j - u_j) = -hu_j + h + O(\varepsilon^2). \quad (4.11)$$

With accuracy up to $\varepsilon^2$, the dynamics of the first chain in (1.5) is defined by the single chain system (1.4) with $F(u) = f(u) - hu + h$ and parameters

$$m_{1,2} = \frac{a}{2} + \sqrt{\frac{a^2}{4} - h}, \quad m_3 = 1. \quad (4.12)$$

Hence, the function $m_3(h)$ is monotonically decreasing. Similarly to (4.7) discussed in the previous subsection and using (4.12), we find the following new behavior:

(d) the kink traveling in the first chain has a greater velocity due to the interchain interaction, $c(h) > c(0)$.

The velocity-coupling dependence for the kink with increased velocity is shown in Fig. 7. The solid curves ($d = 0.15$ and $d = 0.25$) have been obtained from direct simulation of (1.5) and the dashed one ($d = 0.15$) from the perturbation approach using (4.11), (4.12) and (1.4). For $d = 0.15$, one can find the very fine qualitative and quantitative agreement between these curves for small enough $h$, when the perturbation analysis is applicable. Note that the velocity curve ends up at some value of $h$. Similarly to the break of the curves in Fig. 4, this corresponds to the inter-chain synchronization. By contrast with the synchronized kinks, the terminal state here is the “excited” state of both chains, i.e., the kink disappears
as a result of synchronization. Figure 8 illustrates such a kink, whose velocity is increased, by switching on the interaction at \( t_0 \). Note that the dynamics of anti-kinks in system (1.5) is the “opposite” of that described for the kinks. For example, if the kink velocity increases when interacting with the “excited” chain, the anti-kink, if initially excited in the first chain, is slowing down or reversing with the properties (a)-(c). If anti-kink is interacting with the “unexcited” state, its velocity increases with (d)-property. Using the symmetry property of function \( f(u) \), the velocity-coupling dependences for the anti-kink propagation can be easily derived from diagrams of Figs. 4 and 7 by substituting \( a \to 1 - a \).

C. Fronts travelling along steady patterns

Let us consider a kink propagation in the first chain while a steady pattern exists in the second. This situation may occur if the intrachain coupling coefficients are different. In particular, \( d_1(a) \) is taken to provide the propagation, \( c(a) > 0 \) in Fig. 1(c), and \( d_2(a) \) is taken inside \( D_{ch} \) satisfying inequality (3.1) (Sect. III) with \( h = 0 \). For simplicity, from a wealth of possible configurations, we take a periodic steady pattern with quite large spatial scale as shown in Fig. 3(b) (right picture). Then, using (3.3) and (3.5), the coordinates elements are grouped in small neighbourhods near the “excited” \( (O_1) \) and “unexcited” \( (O_0) \) states of the second chain. Using the results of the two previous subsections, we may expect that, for certain values of \( h \neq 0 \), the kink in the first chain would be slowing down while traveling along the “unexcited” elements and speeding-up passing the “excited” elements. This has been verified numerically. We have calculated the instant velocity of the kink during the propagation. Its behavior is shown in Figs. 9(a) and 9(b). The velocity oscillates according to the profile of the steady pattern. The maximum and minimum limit velocity values are defined from the interaction of the kink with the “excited” and “unexcited” chains, respectively, and shown by dashed lines in Fig. 9(b).
V. FRONTS IN ELECTRONIC EXPERIMENTS

A. Experimental set up

Our experiments are carried out on two coupled identical electrical chains. Each of them is composed of N=22 cells (see Figure 10), including a linear capacitance C and a nonlinear resistor $R_{NL}$, whose current-voltage characteristic obeys to the following cubic function

$$I_{NL} = \frac{V}{R_0} \times \left(1 - \frac{V}{\alpha}\right) \times \left(1 - \frac{V}{\beta}\right).$$

Here, $\alpha$ and $\beta$ are the roots of the characteristic, and $R_0$ is a weighting resistor. Intrachain diffusion coupling is assured by linear resistors R, while interchain diffusion coupling between cells referred by the same number j is assured by linear resistors $R'$. Using Kirchhoff laws, we can model the voltages evolutions, namely $U_j$ in chain 1 and $V_j$ in chain 2, by a set of coupled discrete equations:

$$\begin{align*}
\frac{dU_j}{dt} &= \frac{1}{RC} (U_{j+1} + U_{j-1} - 2U_j) - \frac{U_j}{R_0C} (1 - \frac{U_j}{\alpha})(1 - \frac{U_j}{\beta}) - \frac{1}{RC} (U_j - V_j), \\
\frac{dV_j}{dt} &= \frac{1}{RC} (V_{j+1} + V_{j-1} - 2V_j) - \frac{V_j}{R_0C} (1 - \frac{V_j}{\alpha})(1 - \frac{V_j}{\beta}) - \frac{1}{RC} (V_j - U_j).
\end{align*}$$

(5.1)

In addition, the two chains satisfy Neumann boundary conditions.

After normalization, namely setting $u_j = \frac{U_j}{\beta}$, $v_j = \frac{V_j}{\beta}$, $d = R_0 \frac{\alpha}{\beta R_1}$ and $h = R_0 \frac{\alpha}{\beta R'}$, equations (5.1) appear to be an analog simulation of system (1.5).

The state of the system is visualized in a video line, that is, every 64 $\mu$s, the voltage of each cell of chains 1 and 2 is collected as an analog luminance signal, where black corresponds to the state close to $V = 0$ (unexcited state), while white corresponds to the state close to $V = \beta$ (excited state). These definitions were stated in section IV. Using a parallel to serial converter, the resulting serial video output is then mixed with video synchronisation, allowing us to visualise the resulting composite video on a monitor. Therefore, its screen will show the evolution of each cell of both chains vertically, with the time growing toward bottom, such as the whole height corresponds to a local process time of 20 $m$s. For a detailed description of the experimental set up and initial data loading, see [11].
B. Observation of steady states

In order to check the theoretical predictions of section III, summarized in Fig. 3, the components are chosen such as: $R = 100 \, k\Omega$, $R' = 330 \, K\Omega$, $C = 3.3 \, nF$, $\alpha = 0.64 \, V$ and $\beta = 1.45 \, V$. Therefore, $R_0 = 3.2 \, k\Omega$ and the parameters appearing in system (1.5) are fixed to be: $d = 0.014$, $h_j = h = 0.004$, and $a = \frac{\alpha}{\beta} = 0.44$. Note that these parameters verify condition (3.1). The initial conditions, loaded in the two chains, consist of arbitrary sequences of voltages between 0 and $\beta$, that is, respectively, 0 and 1 after normalization. From initial conditions shown in Fig. 11(a), the real system evolves versus time and gives final voltage profiles, as shown in Fig. 11(b), revealing that a steady state is reached for each lattice. These figures confirm that final voltages of every cell of both chain belong to regions $\Omega^0_i$ or $\Omega^1_i$ (see section III and Fig. 3).

C. Front waves slowing down, stopping and about-turn

In this part, we intend to check experimentally the theoretical predictions of section IV.A. The electrical components keep the previous values, except C, which is now 10 $nF$, and intracoupling resistors R, now set to 5.6 $k\Omega$, then $d = 0.25$. In addition, the resistor $R'$ will act as a parameter controlling the interchain coupling coefficient $h$. We consider, first, the case without interchain coupling, that is $h = 0$, and study the propagation of a kink in the first chain, starting from initial conditions where the voltage of the six first cells of chain 1 corresponds to the excited state (white color). The voltage of all other cells, including the cells of chain 2, corresponds to the unexcited state (black color). The correspondent screens (see Fig. 12) show then a propagating front from left to right on the first chain, while nothing occurs in the second chain, that is, voltage of every cell of chain 2 remains in the unexcited state. Note that the chains components values lead for both lattices to an intrachain coupling coefficient $d$ slightly larger than $d^*$, the critical value of propagation failure, whose estimation is given by the inequalities (3.1) with $h = 0$. 

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First, for small value of $h$ (see Fig. 13 with $h = 0.0043$), the kink velocity in the first chain becomes lower than in the case without intercoupling, that is $h = 0$ (Fig. 12), while the behavior of the second chain seems to be unaffected. Comparison between Fig. 13 and Fig. 5 shows a good agreement between theoretical and experimental behaviors. Increasing now $h$, we obtain the screens of Fig. 14, where the front wave is stopped in chain 1, while cells voltages in chain 2 stay very close to zero, that is in the unexcited state (see region $V^0(\gamma_0)$ in Fig. 2(b)). For larger values of $h$, Fig. 15 shows an about-turn of the front wave propagating in chain 1, that is, it propagates now backwards; however, cells of chain 2 still stay in region $V^0(\gamma_0)$ (see also Fig. 2(b) and 6). Finally, for larger values of $h$, as represented in Fig. 16, a kink arises in chain 2, synchronized with the kink of chain 1.

All these experimental results are summarized in Fig. 4, and can be compared with the simulation predictions (continuous line) previously given. One can observe the qualitatively similar behavior between experimental results and simulation predictions. Note however that for practical reasons:

(i) Quantitatively, the experimental crosses of Fig. 4 are not exactly superimposable to the simulation curve, because of different experimental defects: uncertainties on components values, current-voltage characteristic of $R_{NL}$ which differs from the cubic law. For these reasons, the crosses do not exactly agree with the simulation predictions (solid curve) for $d = 0.25$, especially they show a zero-velocity part as predicted theoretically in Fig. 4 for smaller values of $d$ ($d = 0.15$). This discrepancy can be mainly imputable to uncertainties concerning the intrachain coupling coefficient $d$.

(ii) Experimentally, we cannot change all resistors $R'$ for all cells at the same time $t_0$, as it is possible in simulations (see Figs. 5, 6). Thus, the video screens of Figs. 12, 13, 14, 15 and 16, only start after $t_0$.

However, the agreement between the theoretical behavior, for $t > t_0$, and the experimental one is quite satisfactory. In addition, the main features concerning the behavior of the system, that is stopping, about-turn and synchronization of front waves when the intracoupling
increases, correspond to theoretical predictions.

D. Speeding up front waves

This part is devoted to experimental investigations related to section IV.B. Parameters of the system are the same as in in section VI.C, that is, \( a = 0.45 \), \( R_0 = 3.2 \, k\Omega \), \( C = 10 \, nF \) and \( d = 0.25 \). Decreasing resistors \( R' \) step by step, we control the intercoupling parameter \( h \), and measure its effects on the kink propagation. Namely, initial conditions are set as following (see Fig. 17, where \( h=0.014 \) for example): for chain 1, the five first cells are in the excited state 1 (white color), while the others are in the unexcited state (black color); for chain 2, all cells are in the excited state, so they belong to region \( V^1(\beta_0) \) on Fig. 2(b). As the time increases, Fig. 17 shows that the kink propagates now faster in chain 1, with respect to the case \( h = 0 \) (see Fig. 12). This behavior is very similar to the one predicted by simulation and presented in Fig. 8, corresponding to the same parameters, after the change of \( h \) at \( t_0 \).

The evolution of kink velocity \( c \) versus control parameter is drawn in Fig. 7, where the crosses represent experimental results. Although a small quantitative discrepancy between these experimental results and theoretical predictions (continuous lines) is observed, qualitative behaviors are in good agreement, showing that, in real experimental systems, coupling a chain with a second excited one can speed up the front wave.

E. Modulation of kink velocity

In this section, we propose to propagate a kink in the first chain, while a steady state pattern is present in the second one. This case was theoretically considered in section IV.C. Parameters of the system are fixed such as: \( a = 0.44 \), \( R_0 = 3.2 \, k\Omega \), \( R' = 330 \, k\Omega \) \( h = 0.0043 \), but the intracoupling resistors \( R \) are now different along chain 1: \( R = R_1 = 3.3 \, k\Omega \), and along chain 2: \( R = R_2 = 150 \, k\Omega \). These values lead to intracoupling coefficient \( d_1 = 0.43 \) in chain 1, and \( d_2 = 0.0094 \) in chain 2.
Initial conditions for both chains are represented on oscillograms of Figs. 18(a) and (b): only the six first cells of chain 1 are in the excited state, while cells of chain 2 exhibit a turing like pattern (see section III). Evolution versus time of these initial conditions shows (see Fig. 18(c)) a propagation of a front wave in chain 1, but whose instantaneous velocity appears to be not constant: the velocity is yet modulated, with transient values successively larger and smaller than the averaged one. Note that the cell voltages in chain 2 reach quickly a steady state pattern (Fig. 18(d)), which corresponds to a periodic rectangular window, and that low values (resp. large) of the kink velocity in chain 1 correspond to the unexcited cells in chain 2 (resp. excited). Fig. 18(c) being an experimental counterpart of Fig. 9(a), obtained by numerical calculations, a good agreement between these results is observed.

VI. CONCLUSION

In this paper we have studied the dynamics of two coupled discrete Nagumo chains. Unlike to the case of a single discrete Nagumo chain, where no control exists concerning wave front propagation and its unique velocity, we have shown that the coupling coefficient $h$ is very efficient to discriminate different behaviors of the front waves in both coupled chains. We have intensively studied the case of an homogeneous coupling between the chains and shown that an initial front wave introduced in chain 1 can either be slowed down, stopped or “reversed”, according to the strength of the coupling, if chain 2 is initially in the vicinity of the unexcited state. On the other hand, the initial front wave in chain 1 can be speeded up, if chain 2 is initially in the vicinity of the excited state. All these properties have been verified on an experimental electrical system composed of two coupled chains of 22 cells. Our experiments confirm, in particular, the possibility of controlling the dynamics of front waves in coupled chains. This may lead to better understanding natural phenomena observations in a wide class of systems possessing wave front solutions, in particular in models of coupled neural fibers models [2].

Furthermore, extending this study to 2D coupled systems could offer interesting applications.
in the field of image processing. In particular, although replication to another lattice of an image initially stored in a first one has been already studied [30,31], it may be useful to control this replication by means of the interlattices coupling. Moreover, mathematical morphology or contour detection [32] of an initial image would be improved if a coupling coefficient between two lattices could be used to locally control contours propagation, giving either image erosion or dilatation.

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REFERENCES


FIGURE CAPTIONS

Fig. 1 Dependence of the velocity of possible front waves on parameter $m_2$ for $d = 0.05$ in the chain (1.4) with $N = 50$. The curve originating at $m_2 = 0$ with $c(m_2) > 0$ corresponds to kinks and with $c(m_2) < 0$ to anti-kinks.

Fig. 2 (a) Qualitative representation of $J$-set. (b) Qualitative representation of regions $V^\circ(\gamma_0)$ and $V^1(\delta_0)$.

Fig. 3 Formation of steady pattern in both coupled chains described by system (1.5). (a) Initial conditions. (b) Terminal patterns. Parameter values: $a = 0.4, d = 0.01, h = 0.01$ (arbitrary units).

Fig. 4 Velocity-coupling dependences (solid curves) for the kink interacting with “unexcited” state in the second chain. Parameter values: $N = 50, a = 0.45$. Dashed curve shows the dependence $c(h)$ obtained with perturbation analysis (4.5)-(4.7) for $d = 0.15$. The solid lines represent numerical simulation with $d = 0.15$, and $d = 0.25$ (bold line). Experimental results are represented by crosses in the case $d = 0.25 \pm 0.02$.

Fig. 5 Space-time plot of kink propagation in system (1.5). The inter-chain interaction is switched on at time instant $t_0$. The dynamics of the $u_j, v_j$-components is shown by the levels of grey (the scale is not shown) with the white color corresponding to the “excited” state and black color to the “unexcited” state. The kink corresponds to the white-black interface. The light grey at the right picture shows the small perturbation of the second chain. Parameter values: $N = 50, a = 0.45, d = 0.15, h = 0.01$.

Fig. 6 Kink reversing in system (1.5). Parameter values: $N = 50, a = 0.45, d = 0.15, h = 0.03$.

Fig. 7 Velocity-coupling dependences (solid curves) for the kink in chain 1 interacting with “excited” state in the second chain. Dashed curve shows the dependence $c(h)$ obtained with perturbation analysis (4.11)-(4.12), with $d = 0.15$. Other parameter
values: $N = 50, a = 0.45$. The solid lines represent numerical simulation with $d = 0.15$, and $d = 0.25$ (bold line). Experimental results are represented by crosses in the case $d = 0.25 \pm 0.02$.

Fig. 8 Kink speeding up in the result of interaction with the “excited” state in chain 2.
Parameter values: $N = 50, a = 0.45, d = 0.15, h = 0.02$.

Fig. 9 Kink propagating along the steady periodic pattern in the second chain (Fig. 3 (b), right picture). (a) Modulation of the kink velocity in space-time plot $(j,t)$ for chain 1. (b) Instantaneous velocity, $c_i(j) = \frac{1}{\tau_j}$ where $\tau_j$ is the time of the front to come from $(j - 1)th$ to $jth$ element, during the propagation. Dashed curves corresponds to the homogeneous states $O_1$ (upper line) and $O_0$ (lower line) in the second chain. Parameter values: $N = 50, a = 0.4, d_1 = 0.1, d_2 = 0.01, h = 0.01$.

Fig. 10 Sketch of the experimental coupled electrical lattice.

Fig. 11 Steady states or spatial patterns on both coupled chains. (a) initial condition, (b) final state. Parameter values after normalization are $a = 0.44 \pm 0.01, h = 0.0043 \pm 0.0003, d = 0.014 \pm 0.001$ (all in arbitrary units), while $C = 3.3 \text{ nF}$.

Fig. 12 Kink propagation on the first chain without interchain coupling. The second chain is in unexcited state. $a = 0.45 \pm 0.01, h = 0, d = 0.25 \pm 0.02, C = 10 \text{ nF}$.

Fig. 13 The kink velocity of kink decreases in the first lattice, the second one staying in unexcited state. $a = 0.45 \pm 0.01, h = 0.0043 \pm 0.0002, d = 0.25 \pm 0.02, C = 10 \text{ nF}$.

Fig. 14 Stop propagation of kink for $a = 0.45 \pm 0.01, h = 0.030 \pm 0.002, d = 0.25 \pm 0.02$, $C = 10 \text{ nF}$.

Fig. 15 The kink velocity becomes negative when $a = 0.45 \pm 0.01, h = 0.043 \pm 0.002$, $d = 0.25 \pm 0.02, C = 10 \text{ nF}$.
Fig. 16 Instantaneous synchronization of kinks in both lattices for $a = 0.45 \pm 0.01$, $h = 0.094 \pm 0.005$, $d = 0.25 \pm 0.02$, $C = 10 \ nF$.

Fig. 17 When the second chain is initially in excited state, the velocity of wave front in the first chain is increased for $h \neq 0$. Parameters values: $a = 0.45 \pm 0.01$, $h = 0.014 \pm 0.001$, $d = 0.25 \pm 0.02$, $C = 10 \ nF$.

Fig. 18 Velocity modulation of kink in chain 1. Starting from initial conditions $(a)$ and $(b)$, the kink on chain 1 undergoes velocity fluctuations $(c)$ related to the steady state pattern existing in chain 2 $(d)$. $a = 0.44 \pm 0.01$, $h = 0.0043 \pm 0.0003$, $d_1 = 0.43 \pm 0.03$, $d_2 = 0.0094 \pm 0.0008$, $C = 10 \ nF$. 

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