Propagation failure induced by coupling inhomogeneities in a nonlinear diffusive medium

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Abstract

Kink propagation failure induced by coupling inhomogeneities in a Nagumo lattice is investigated. Considering the case of weak couplings, we define analytically and numerically the coupling conditions leading to the pinning of the kink.

Key words: Propagation failure, Inhomogeneities, Critical coupling, Nonlinear diffusive media.

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1 Introduction

In recent years, the dynamics of kinks or traveling wave fronts in strongly dissipative or reaction-diffusion systems has attracted considerable attention (1; 2; 3; 4). In particular, transport processes in living cells, neural networks, physiological systems and phenomena of cardiac arrhythmias like reentry and fibrillation are modelled by discrete reaction-diffusion equations supporting kink solutions. In such systems, and contrary to continuous models, there exist particular effects intrinsically related to the discrete character of the considered medium. Among them, is the well known wave propagation failure phenomenon, which appears when the intrachain coupling becomes lower than a non zero critical value (5; 6; 8; 9). More surprisingly, propagation failure may also occur in a discrete medium with intrachain coupling $D_1$ due to a coupling inhomogeneity $D_2$ greater than $D_1$. Indeed, very recently, Wang and Rudy investigated action potential propagation in inhomogeneous cardiac tissues, using as a model of structural inhomogeneity, a nonlinear lattice consisting of the junction of two sub-lattices with different couplings (11). In particular, they...
observed that propagation fails if the intercellular coupling $D_2$ in the second sub-lattice exceeds a critical value $D_{sup}^*$. More generally, this means that, considering a traveling wave front in a lattice with small and inhomogeneous intrachain coupling, propagation exists if the coupling inhomogeneities belong to an interval defined by two critical coupling values $D_{inf}^*$ and $D_{sup}^*$. Until now, all the studies have been devoted to the determination of $D_{inf}^*(5; 6; 7; 9)$, that is, the non zero coupling value under which propagation failure occurs. However, to our knowledge, there exists no theoretical determination of the critical value $D_{sup}^*$ over which propagation fails.

The aim of this letter is to propose a theoretical determination of the critical coupling value $D_{sup}^*$ over which no propagation is possible.

In order to investigate the effects induced by local coupling discontinuities in a discrete diffusive lattice, as in (11), we consider the same case of a unique discontinuity existing at the junction of two homogeneous nonlinear diffusive discrete Nagumo type chains $M_1$ and $M_2$ with different couplings, respectively $D_1$ and $D_2$. Our two complementary theoretical approaches, presented in the main section of this paper, allow to determine a domain of the critical coupling $D_{sup}^*$ in good agreement with the numerical simulations results.

2 Theoretical study

Let us consider the following set of nonlinear diffusive equations modelling the evolution of the two connected Nagumo-type media $M_1$ and $M_2$ presented in Fig.1.(a).

$$\begin{align*}
\frac{du_n}{dt} &= D_1(u_{n-1} - 2u_n + u_{n+1}) + f(u_n) & \text{for } n < m \\
\frac{du_n}{dt} &= D_2(u_{n-1} - 2u_n + u_{n+1}) + f(u_n) & \text{for } n > m \\
\frac{du_n}{dt} &= D_1(u_{n-1} - u_n) + D_2(u_{n+1} - u_n) + f(u_n) & \text{for } n = m,
\end{align*}$$

(1)

where the function $f(u_n) = -u_n(u_n-a)(u_n-1)$ represents the cubic on-site nonlinearity, and $m$ is the interface site.

The two first equations of system (1) describe the evolutions of media $M_1$ and $M_2$, respectively, whereas the third one is related to the interface site.

From a physical point of view, system (1) can also model an overdamped chain of harmonically coupled particles of mass $M$ lying in a double well on site potential $U(u_n)$ (see Fig. 1.(b)), where the inertia term is neglected. The function $f(u_n) = -dU(u_n)/du_n$ represents a force deriving from the potential $U(u_n)$ and the diffusive coefficients in each medium are expressed by $D_1 = k_1/\lambda$ and $D_2 = k_2/\lambda$, $k_1$ and $k_2$ being the coupling strengths in the
two lattices, respectively and \( \lambda \) the coefficient of the friction force.

\[
\begin{align*}
D_1(1 - u_m) - D_2 u_m + f(u_m) &= 0 \, \quad (2)
\end{align*}
\]

Finally, substituting the \( f(u_m) \) cubic expression in (2) leads to the following polynomial

\[
P(u_m) = u_m^3 - (1 + a) u_m^2 + (a + D_1 + D_2) u_m - D_1 = 0. \, \quad (3)
\]

In figure (2), for a given value of \( D_1 \) and \( a \), we plot \( P(u_m) \) for three values of \( D_2 \). From a physical point of view, as the interface site \( m \) corresponds to a position \( u_m < a \) (see Fig. 1), the rightmost root is not appropriate. Therefore, for a given couple \( (a, D_1) \), the existence of solutions depends only on \( D_2 \). The critical value \( D_{\text{sup}}^* \), at which stationary solutions cease to exist, is obtained when the maximum of \( P(u_m) \) becomes null (see Fig. 2).

This corresponds to the case:

\[
\delta = \frac{1}{3} (a + D_1 + D_2) - \frac{1}{9} (1 + a)^2 + \frac{1}{6} (3D_1 - (1 + a)(a + D_1 + D_2)) + \frac{1}{27} (1 + a)^3 = 0,
\]

Fig. 1. Schematic representation of the two connected nonlinear diffusive media (a) and the equivalent mechanical lattice (b) with the interface site \( m \) corresponding to a position \( u_m < a \).

Fig. 2. Representation of the polynomial \( P(u_m) \) obeying to relation (4) for three different values of \( D_2 \): a) \( D_2 = 0.05 \); b) \( D_2 = 0.038 \); c) \( D_2 = 0.029 \). Parameters are \( D_1 = 0.03 \) and \( a = 0.3 \).
where $\delta$ is the super discriminant of $P(u_m)$. Using an implicit plot of

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This equation admits no solutions in $u_m$ if its discriminant is negative, that is:

$$\Delta = (a \gamma - D_1 - D_2)^2 + 4 \gamma D_1 < 0. \quad (7)$$

Then, the critical coupling value $D_2 = D_{sup}^*$, under which the kink propagation is possible through the interface, is expressed by:

$$D_{sup}^* = D_2 = \sqrt{D_1(1 - a/2)} - D_1 - a + a/2, \quad (8)$$

which corresponds to curve (b) in Fig. 3. Note that, the numerically obtained data lie between the two theoretical curves corresponding to our approaches, giving evidence that neither of our two methods are sufficiently sharp to determine precisely the front dynamics at the interface. Nevertheless, our results allow to determine a domain of the critical coupling $D_{sup}^*$ respectively over which and under which the kink does not or does overcome the interface.

In order to complete the propagation failure effect induced by coupling inhomogeneities in a Nagumo lattice, we have also reported in Fig. 3 the critical coupling $D_{inf}^*$ (9) under which standard propagation failure occurs (curve c). This leads to the several propagation and no propagation domains pointed out in Fig. 3.

3 Conclusion

In summary, we have studied the kink dynamics at the interface of two discrete Nagumo chains of different couplings. We have shown that for a given coupling $D_1$ in the first lattice, there exists a range $[D_{inf}^*, D_{sup}^*]$ of coupling $D_2$ in the second lattice allowing the kink to overcome the interface. In particular, an analytical determination of $D_{sup}^*$ is presented using two complementary methods. These analytical predictions are then in good agreement with numerical simulations and confirm the propagation failure induced by an abrupt change of intercellular coupling observed by Wang and Rudy in cardiac tissue. Note that, in the case of a kink propagating in a homogeneous lattice with coupling $D_1$ and presenting only one localized coupling defect, that is a local coupling $D_2 \geq D_{sup}^*$, the kink propagation fails too. This result could be extended to other coupling distributions, periodic or nor. Finally, and more generally, our results could be useful to better understanding inhomogeneities effects in nonlinear physical systems with...
intrinsic discrete structure.

References