PINNING OF A KINK IN A NONLINEAR DIFFUSIVE MEDIUM WITH A GEOMETRICAL BIFURCATION: THEORY AND EXPERIMENTS

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Abstract. We study the dynamics of a kink propagating in a Nagumo chain presenting a geometrical bifurcation. In the case of weak couplings, we define analytically and numerically the coupling conditions leading to the pinning of the kink at the bifurcation site. Moreover, real experiments using a nonlinear electrical lattice confirm the theoretical and numerical predictions.

1 Introduction

Since the past two decades, a growing interest has been devoted to the dynamics of kinks in strongly dissipative or reaction-diffusion systems [Murray 1989; Nagumo et al., 1962; Keener 2000; Comte et al., 1999; Ferreira et al., 2002]. Indeed, discrete reaction-diffusion equations supporting kink solutions are widely used to describe the excitation spread in a variety of systems in physics, biology or chemistry [Murray 1989; Scott 1999]. In such discrete media, like chains of coupled chemical reactors [Laplante and Erneux, 1992], myelinated nerve fibers or cardiac tissues [deCastro et al., 1992], there exist effects directly imputable to the discrete character of the considered medium. Among them, is the well known wave propagation failure phenomenon occurring when the intra-chain coupling becomes lower than a non zero critical value [Keener 1987; Erneux and Nicolis, 1993; Mitkov et al., 1998; Comte et al., 2001; Kladko et al., 2000; Fath 1998, Báscones et al., 2002].

Recent studies of action potential propagation in cardiac tissues have shown, that propagation in a discrete medium could also fail owing to an abrupt increase of the intercellular coupling [Wang and Rudy, 2000]. Indeed, considering the junction of two homogeneous nonlinear diffusive Nagumo-type sub-lattices with respective couplings $D_1$ and $D_2$, the kink initiated in the first medium will overcome the junction if the coupling $D_2$ belongs to an interval $[d_{\text{inf}}, d_{\text{sup}}]$ [Morfu et al., 2002; Mornev 1991; Morfu et al., 2002]. In a discrete diffusive medium, one might also wonder how the presence of branches at a bifurcation site may quantitatively affect the propagation conditions through this site. This leads us to investigate in the present paper, how a kink propagating in a main fiber $F_1$ separating in two
fibers (or branches) $F_2$ and $F_3$ with respective intercellular couplings $D_1$, $D_2$ and $D_3$, will overcome the bifurcation site.

In the particular case of two identical fibers $F_2$ and $F_3$ with couplings $D_2 = D_3 = D$, we show that the propagation is possible through the bifurcation site for $D \in [D_{inf}^*, D_{sup}^*]$.

The paper is organized as follows: in the next section, using discrete Nagumo-type fibers, we analyze theoretically and numerically the coupling conditions inducing the pinning of the kink at the bifurcation site. These predictions are then confirmed, in section 3, by experimental results obtained with an equivalent electrical lattice. We finally give some concluding remarks in the last section.

## 2 Theoretical and numerical study

The set of nonlinear diffusive equations modelling the evolution of a Nagumo chain $F_1$ presenting a bifurcation site $m$ with two branches $F_2$ and $F_3$ (figure 1.a) is expressed, in the general case, under the following normalized form

$$
\begin{align*}
\frac{dv_{1,n}}{dt} &= D_1(v_{1,n-1} + v_{1,n+1} - 2v_{1,n}) + f(v_{1,n}) \text{ for } n < m, \\
\frac{dv_{2,n}}{dt} &= D_2(v_{2,n-1} + v_{2,n+1} - 2v_{2,n}) + f(v_{2,n}) \text{ for } n > m, \\
\frac{dv_{3,n}}{dt} &= D_3(v_{3,n-1} + v_{3,n+1} - 2v_{3,n}) + f(v_{3,n}) \text{ for } n > m, \\
v_{1,m} &= v_{2,m} = v_{3,m} \\
\frac{dv_{1,m}}{dt} &= D_1(v_{1,m-1} - v_{1,m}) + D_2(v_{2,m+1} - v_{1,m}) + D_3(v_{3,m+1} - v_{1,m}) + f(v_{1,m}),
\end{align*}
$$

(1)

where the cubic on-site nonlinearity is provided by $f(v_{i,n}) = -v_{i,n}(v_{i,n} - a)(v_{i,n} - 1)$ with the index $i \in \{1, 2, 3\}$.

The three first equations of system (1) describe the evolutions of each fiber, respectively $F_1$, $F_2$, and $F_3$, while the last two ones are related to the bifurcation site $m$.

As in this paper the two fibers $F_2$ and $F_3$ are identical, the system (1) reduces to

$$
\begin{align*}
\frac{dv_{1,n}}{dt} &= D_1(v_{1,n-1} + v_{1,n+1} - 2v_{1,n}) + f(v_{1,n}) \text{ for } n < m, \\
\frac{dv_{i,n}}{dt} &= D(v_{i,n-1} + v_{i,n+1} - 2v_{i,n}) + f(v_{i,n}) \text{ for } n > m \text{ and } i \in \{2, 3\}, \\
v_{1,m} &= v_{2,m} = v_{3,m} \\
\frac{dv_{1,m}}{dt} &= D_1(v_{1,m-1} - v_{1,m}) + D(v_{2,m+1} - 2v_{1,m} + v_{3,m+1}) + f(v_{1,m}).
\end{align*}
$$

(2)
From a physical point of view, the system (2) can also model an overdamped chain of harmonically coupled particles of mass $M$ lying in a double well on-site potential $\phi(v_{i,n})$ (see 1.b), where the inertia terms are neglected. The function $f(v_{i,n}) = -d\phi(v_{i,n})/dv_{i,n}$ represents a force deriving from the potential $\phi(v_{i,n})$ and the diffusive coefficients in each medium are expressed by $D_1 = k_1/\lambda$, $D = k/\lambda$, $k_1$ and $k$ being the coupling strengths in the main fiber and the two branches respectively, whereas $\lambda$ is the friction force coefficient.

For a given coupling $D_1$, we propose to determine analytically the critical coupling $D = D^*_{sup}$ over which a kink spreading in the chain $F_1$ cannot overcome the bifurcation site $m$. Note that the lower critical value $D^*_{inf}$ can be estimated by the critical coupling $D^*$ under which propagation failure occurs in a homogeneous medium [Keener 1987; Erneux and Nicolis, 1993; Comte et al., 2001].

As our study deals with weak couplings, we can assume, as in [Kladko et al., 2000; Morfu et al., 2002], that when the kink is pinned at the bifurcation site $m$, only this site experiences nonlinearity while the site $m-1$ of the main chain $F_1$ and the sites $m+1$ of each branch $F_2$ and $F_3$ are respectively close enough to the potential minima 1 and 0, that is $v_{1,m-1} = 1$, $v_{2,m+1} = 0$ and $v_{3,m+1} = 0$.

Under these conditions, setting $v = v_{1,m}$, the system (2) reduces to

$$D_1(1-v) - 2Dv + f(v) = 0. \quad (3)$$

For a non pinning of the initial travelling front wave at the junction, let us point out that it is necessary for the front site $m$ (bifurcation site), to pass the energy barrier (with maximum height $\Delta \phi$) in $v = a$, separating the two potential minima (see Fig. 1). Therefore, as in [Comte et al., 2001] the double well on site potential $\phi(v)$ can be replaced on the interval $[0, a]$ by an equivalent third order polynomial respecting the same extrema ($v = 0$ and $v = a$), and the same barrier height $\Delta \phi$ (continuous line in Fig. 2).

Then, the nonlinear force or function $f(v)$ can be replaced on $[0, a]$ by

$$g(v) = \gamma v (v-a), \quad (4)$$

where $\gamma = \frac{a}{2} - 1$ is obtained by identification of the barrier height.

Substituting (4) in eq.(3), provides the following expression:

$$\gamma v^2 - (a \gamma - D_1 - 2D) v - D_1 = 0. \quad (5)$$

This equation admits no solutions in $v$ if its discriminant is negative, that is:

$$\Delta = (a \gamma - D_1 - 2D)^2 + 4 \gamma D_1 < 0. \quad (6)$$
Therefore the critical coupling value $D = D_{\text{sup}}^*$, under which the kink propagation is possible through the bifurcation site $m$, is expressed by:

$$D_{\text{sup}}^* = D = -D_1/2 + a^2/4 - a/2 + 1/2\sqrt{4D_1 - 2aD_1}. \quad (7)$$

In order to validate the analytical expression (7) of $D_{\text{sup}}^*$, we have performed numerical simulations using a fourth order Runge-Kutta algorithm with an integrating time step $dt = 0.001$. The coupling coefficient $D_1$ being fixed, we initiate a kink propagating in the main fiber $F_1$. Proceeding by dichotomy, we simulate the system for different couplings values $D = D_2 = D_3$ of the two branches $F_2$ and $F_3$ to obtain the two critical values $D_{\text{inf}}^*$ and $D_{\text{sup}}^*$ defining the propagation conditions through the bifurcation site $m$.

As one can see on figure 3, provided that the coupling coefficient is weak enough, our analytical approach is in good agreement with the numerical results. Note that, for larger couplings, our theoretical predictions provide nevertheless a fairly good estimation of the region where propagation is possible.

### 3 Real experiments

In order to complete the propagation failure effect induced by a geometrical bifurcation, we also present experimental results obtained with the nonlinear electrical lattice [Nekorkin et al., 2001] of the figure 4.

In the experiments, the first medium $F_1$ is realized with 24 elementary cells, resistively coupled by linear resistors $R_1$ while each of the two branches $F_2$ and $F_3$ are obtained with 12 others elementary cells coupled by linear and adjustable resistors $R_2 = R_3$. The bifurcation site is then located at the 24th cell, that is $m = 24$. Each elementary cell is constructed with a linear capacitor $C$ in parallel with a nonlinear resistor $R_{NL}$ whose current-voltage characteristic obeys to the following cubic law

$$I(V) = V(V - \alpha)(V - \beta)/(R_0\alpha\beta), \quad (8)$$

with $R_0 = 3.2 \, k\Omega$ and $a = \alpha/\beta = 0.3$.

Using Kirchhoff laws, we obtain straightforwardly the set of differential equations corresponding to a
Nagumo chain with a geometrical bifurcation at the site $m = 24$:

$$
\begin{align*}
\frac{dV_{1,n}}{dt} &= \frac{1}{R_1 C} (V_{1,n-1} + V_{1,n+1} - 2V_{1,n}) - \frac{V_{1,n}}{R_0 C} (1 - \frac{V_{1,n}}{\alpha}) (1 - \frac{V_{1,n}}{\beta}) \quad \text{for } n < m, \\
\frac{dV_{2,n}}{dt} &= \frac{1}{R_2 C} (V_{2,n-1} + V_{2,n+1} - 2V_{2,n}) - \frac{V_{2,n}}{R_0 C} (1 - \frac{V_{2,n}}{\alpha}) (1 - \frac{V_{2,n}}{\beta}) \quad \text{for } n > m, \\
\frac{dV_{3,n}}{dt} &= \frac{1}{R_3 C} (V_{3,n-1} + V_{3,n+1} - 2V_{3,n}) - \frac{V_{3,n}}{R_0 C} (1 - \frac{V_{3,n}}{\alpha}) (1 - \frac{V_{3,n}}{\beta}) \quad \text{for } n > m, \\
\frac{dV_{1,m}}{dt} &= \frac{1}{R_1 C} (V_{1,m-1} - V_{1,m}) + \frac{1}{R_2 C} (V_{2,m+1} - V_{1,m}) \\
&\quad + \frac{1}{R_3 C} (V_{3,m+1} - V_{1,m}) - \frac{V_{1,m}}{R_0 C} (1 - \frac{V_{1,m}}{\alpha}) (1 - \frac{V_{1,m}}{\beta}).
\end{align*}
$$

After normalization, namely setting $v_{i,n} = \frac{V_{i,n}}{\beta}$, $D_i = R_0 \frac{\alpha}{\beta R_i}$ with $i \in \{1, 2, 3\}$, an analog simulation of the system (1) is realized with the electrical nonlinear lattice.

Initiating a kink from a Heaviside-type initial condition loaded in the first medium $F_1$, we investigate versus the resistor $R_2 = R_3$ if the kink overcomes the bifurcation site $m = 24$ and spreads in each branch $F_2$ and $F_3$. For a given resistor value $R_1$, all the resistor $R_2$ and $R_3$ are adjusted to their maximum value (the very small coupling case) in order to set the two sub-lattices $F_2$ and $F_3$ in standard propagation failure regime ($D_2 = D_3 < D_{inf}^{*}$). Therefore, the kink is pinned at the bifurcation site.

Then, these resistors are simultaneously decreased until the critical value $R_{inf}$ (corresponding to $D_{inf}^{*}$) is reached allowing the kink to overcome the bifurcation site as shown in the spatiotemporal diagrams of figure 5 providing the evolution of each fiber, $F_1$, $F_2$ and $F_3$ respectively. Starting from the initial condition represented above the dotted line in gray scale, the kink initiated in the main fiber $F_1$ overcomes the bifurcation site and spreads with the same velocity in each branch $F_2$ and $F_3$ since they are identical.

Decreasing now slightly $R_2$ and $R_3$ (namely increasing $D_2 = D_3$) to the second critical value $R_{sup}$ (corresponding to $D_{sup}^{*}$), involves once again the pining of the kink at the bifurcation site.

This leads to the several propagation failure domains of figure 3 where the experimental results are compared to the numerical ((+)) signs and the theoretical predictions. Despite some discrepancies mainly imputable to the component uncertainties, the propagation failure behavior induced by the presence of a geometrical bifurcation is however confirmed experimentally.
4 Conclusion

We have studied the kink dynamics in a Nagumo chain presenting a geometrical bifurcation with two branches. In the discrete case, we have shown analytically that for a given coupling $D_1$ of the main fiber, there exists a range $([D_{inf}^*, D_{sup}]^*)$ of coupling $D_2 = D_3$ of the two branches allowing the kink to overcome the bifurcation site. These theoretical results are confirmed by numerical simulations and by real experiments using a nonlinear electrical lattice. Our study could be extended to multiple branches, identical or not, where one could expect that the propagation domain decreases with the number of added branches. In particular, considering the propagation in the reverse direction with a kink initiated in one or the two branches, an extension of this work could explain how a logical “OR” function could be implemented with this kind of Nagumo chain. Moreover, we believe that this work could be interesting in better understanding phenomena in the context of neuro or cardiophysiology.
References


**Figure captions**

**Fig. 1**: Schematic representation of a nonlinear diffusive medium with a geometrical bifurcation (a) and its equivalent mechanical lattice (b).

**Fig. 2**: Graphical representation of the substrate potential (dotted line) for the specific case $a = 0.3$ and its locally equivalent third order polynomial (bold line) on the interval $[0, a]$. $\Delta V$ is the energy barrier.

**Fig. 3**: Determination of propagation failure domains induced by a geometrical bifurcation. The solid line (a) corresponds to the theoretical determination of $D_{sup}^*$ according to relation (7), while the corresponding results obtained by numerical simulations are represented by the $o$ signs. The crosses with their uncertainties represent experimental results obtain with the reaction-diffusion electrical line. Curve (b) gives the limit $D_{inf}^*$ under which propagation fails. $a = \alpha/\beta = 0.3$.

**Fig. 4**: Sketch of the nonlinear electrical lattice with two branches at the bifurcation site $m = 24$.

**Fig. 5**: Experimental evolution of the main fiber $F_1$, and the two branches $F_2$ and $F_3$. $m = 24$, $a = \alpha/\beta = 0.3 \pm 0.01$, $C = 33nF$, $R_1 = 10k\Omega$, $R_2 = R_3 = 15k\Omega$, that is $D_1 = 0.096 \pm 0.005$, $D = 0.064 \pm 0.004$. White color corresponds to the excited level $\beta = 1.6 V$, while black corresponds to unexcited state $0 V$. 


Figure 1: S Morfu et al.
Figure 2: S. Morfu et al.
\[ D_3 = D_2 = D (\text{arb. units}) \]

Figure 3: S. Morfu et al.
Figure 4: S. Morfu et al.
Figure 5: S. Morfu et al.