An algorithm to construct 3D triangles with circular edges

Bertrand Belbis\textsuperscript{1}, Lionel Garnier\textsuperscript{1}, Sebti Foufou\textsuperscript{1,2}

(1) LE2I Lab, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France
bbelbis@u-bourgogne.fr, lgarnier@u-bourgogne.fr
(2) CSE Dept., College of Engineering, Qatar University, P.O. Box 2713, Doha Qatar
sfoufou@u-bourgogne.fr, sfoufou@qu.edu.qa

ABSTRACT

Dupin cyclides are non-spherical algebraic surfaces of degree 4, discovered by the French mathematician Pierre-Charles Dupin at the beginning of the 19th century. A Dupin cyclide has a parametric equation and two implicit equations and circular lines of curvature. It can be defined as the image of a torus, a cone of revolution or a cylinder of revolution by an inversion. A torus has two families of circles: meridians and parallels. There is a third family of circles on a ring torus: Villarceau circles. As the image, by an inversion, of a circle is a circle or a straight line, there are three families of circles onto a Dupin cyclide too. The goal of this paper is to construct, onto a Dupin cyclide, 3D triangles with circular edges: a meridian arc, a parallel arc and a Villarceau circle arc.

Keywords: 3D Triangles, circular edges, Dupin cyclides, rational biquadratic Bézier surfaces.

1 Introduction

In C.A.D, a complex object can be represented by a triangular mesh. This representation gives a lot of freedom for the design, but it needs a large number of topological conditions. The storage and the visualization of complex meshes may be very costly in memory and time, fundamental geometric operations such as intersections may also be very heavy when they are carried on a several complex meshes in the same time.

It would be interesting to have an object formed by 3D triangles which have an algebraic and a parametrical representation. On each triangle no topological information is needed. The parametric and implicit equations of the 3D triangles will be used for shape modeling, and to generate some planar approximations for the visualization purposes. In this work, we propose to computed 3D triangle on a family of algebraic surfaces, called Dupin cyclides. Dupin cyclides are degree 4 algebraic surfaces discovered by the French mathematician Pierre-Charles Dupin in 1822. These surfaces have a number of properties (e.g. circular lines of curvature, principal circles, ... ) that facilitate their use in geometric modeling. The property that lists the families of circles that can be drawn on a Dupin cyclide is of the most importance to the work presented in this paper. Dupin cyclides have three families of circles: the parallels, the meridians and the Villarceau ones. The aim of this work is to build 3D triangles belonging to a Dupin cyclide, where the edges are arcs of parallel, meridian and Villarceau circles.

The paper is organized as follow: in section 2, we recall some definitions and properties of rational quadratic Bézier curves, definitions of rational biquadratic Bézier surfaces and definition and properties of Dupin cyclides. Section 3 presents the proposed algorithm for the construction of 3D triangles on a Dupin cyclide. Section 4 concludes the paper.

2 Background

2.1 Rational Bézier curves and surfaces

Rational quadratic Bézier curves are second degree parametric curves defined by:

\[
\overrightarrow{OM}(t) = \frac{\sum_{i=0}^{2} w_i B_i(t) \overrightarrow{OP}_i}{\sum_{i=0}^{2} w_i B_i(t)}, \quad t \in [0, 1]
\]  

(1)

where \( B_i(t) \) are quadratic Bernstein polynomials defined as:

\[
B_0(t) = (1 - t)^2, \quad B_1(t) = 2t(1 - t) \quad \text{and} \quad B_2(t) = t^2
\]
and for \( i \in \{0, 1, 2\} \), \( w_i \) are weights associated with the control points \( P_i \). For a standard rational quadratic Bézier curve, \( w_0 \) and \( w_2 \) are equal to 1, while \( w_1 \) can be used to control the type of conic defined by the curve [Far93, Far99, Gar07].

As we will modelize circle arcs using rational quadratic Bézier curve in our algorithm, we recall a theorem to compute the weight \( \omega_1 \):

**Theorem 1 Circle defined by two points and its tangents at this points**

Let \( P_0, P_1 \) and \( P_2 \) three non-colinear points. \((P_0P_1)\) and \((P_2P_1)\) are the tangents to the circle \( C \) at \( P_0 \) and \( P_2 \). The center of \( C \) is \( O_0 \) and the radius of \( C \) is \( R \).

Let \( I_1 \) be the middle of \([P_0, P_2]\).

Let \( P_c \) be the perpendicular bisector plane of \([P_0, P_2]\).

Let \( G = \text{bar}\{(P_0, \omega_0), (P_2, \omega_2)\}\) (bar is a notation for barycentre).

The rational quadratic Bézier curve \( \gamma \) of level-headed control points \((P_0, \omega_0), (P_1, \omega_1)\) and \((P_2, \omega_2)\) is an arc of circle if and only if:

\[
\left\| O_0 \gamma \left( \frac{1}{2} \right) \right\| = R \tag{2}
\]

which is equivalent to the equation \( \alpha \omega_1^2 + \beta = 0 \), where \( \alpha \) and \( \beta \) are given by:

\[
\alpha = 4(R^2 - O_0 P_0^2), \quad \beta = \omega^2(R^2 - O_0 G^2)
\]

The positive solution \( \omega_1^+ \) (resp. negative \( \omega_1^- \)) of the solutions

\[
\omega_1^+ = \sqrt{-\beta \over \alpha}, \quad \omega_1^- = -\sqrt{-\beta \over \alpha} \tag{3}
\]

let us design the small (resp. the great) circle arc of the circle \( C \) with \( P_0 \) and \( P_2 \) as bounds.

Let us note that if \( \omega_0 = \omega_2 = 1 \) then the computation of weight \( \omega_1 \) is simplified to:

\[
\omega_1^+ = \cos \left( P_0 \overrightarrow{P_1} P_0 P_2 \right), \quad \omega_1^- = -\cos \left( P_0 \overrightarrow{P_1} P_0 P_2 \right)
\]

Rational biquadratic Bézier surfaces are defined by control points \((P_{ij})_{0 \leq i,j \leq 2}\) and weights \((w_{ij})_{0 \leq i,j \leq 2}\) as:

\[
OM(u,v) = \frac{\sum_{i=0}^{2} \sum_{j=0}^{2} w_{ij} B_i(u) B_j(v) \overrightarrow{P_{ij}}}{\sum_{i=0}^{2} \sum_{j=0}^{2} w_{ij} B_i(u) B_j(v)} \tag{4}
\]

The following theorem for the computation of the barycentric middle curve on a Bézier surface will be used in the Section 3. The proof of this theorem can be found in [GBF09]:

**Theorem 2 Barycentric middle curve**

Let us consider a Bézier surface defined by control points \((P_{ij})_{0 \leq i,j \leq 2}\) and weights \((w_{ij})_{0 \leq i,j \leq 2}\).

Let \( G^+ = \text{bar} \{(P_{00}, \omega_{00}), (P_{11}, 2\omega_{11}), (P_{22}, \omega_{22})\} \) and \( \alpha_i = \omega_{i0} + 2\omega_{i1} + \omega_{i2} \) where \( i \in [0; 2] \).

Let \( G^- = \text{bar} \{(P_{00}, \omega_{00}), (P_{11}, 2\omega_{11}), (P_{22}, \omega_{22})\} \) and \( \alpha_i = \omega_{i0} + 2\omega_{i1} + \omega_{i2} \) where \( i \in [0; 2] \).

If \( \sum_{i=0}^{2} \alpha_i \neq 0 \), the barycentric middle curve

\[
u \mapsto M \left( u, \frac{1}{2} \right)
\]

is a Bézier curve with control points \( (G^i; \alpha_i)_{i \in [0; 2]} \).

If \( \sum_{i=0}^{2} \alpha_i \neq 0 \), the barycentric middle curve

\[
u \mapsto M \left( \frac{1}{2}, v \right)
\]

is a Bézier curve with control points \( (G_i^+; \alpha_i)_{i \in [0; 2]} \).

More details on Bézier surfaces can be found in [Far93, Far99, Gar07]. In the remaining of this paper, we only consider rational Bézier curves and surfaces of degree two to which we refer, for short, by Bézier curves and Bézier surfaces.

### 2.2 Dupin cyclides

Non-degenerate Dupin cyclides, figure 1, have been defined by P. Dupin [Dup22]. A simple definition is: a Dupin cyclide is the image of a torus, a circular cone or a circular cylinder by an inversion [Gar07].

A. R. Forsyth [For12] and G. Darboux [For12, Dar17] have given two equivalent implicit equations:

\[
(x^2 + y^2 + z^2 - \mu^2 - b^2)^2 = 4(ax - c\mu)^2 + 4b^2y^2 \tag{5}
\]

\[
(x^2 + y^2 + z^2 - \mu^2 - b^2)^2 = 4(ec - a\mu)^2 - 4b^2z^2 \tag{6}
\]

in an orthonormal basis \((O, \vec{u}_0, \vec{u}_1, \vec{u}_0)\), where \( O \) is called Dupin cyclide center, and parameters \( a, b \) and \( c \) are related by \( c^2 = a^2 - b^2 \). The parameter \( a \) is always greater than or equal to \( c \).

A parametric equation of a Dupin cyclide is:

\[
\Gamma_d(\theta, \psi) = \begin{pmatrix}
\mu(c - a\cos \theta \cos \psi) + b^2 \cos \theta \\
\frac{a - c \cos \theta \cos \psi}{b \sin \theta \times (a - c \cos \psi)} \\
\frac{b \cos \theta \times (c \cos \theta - \mu)}{a - c \cos \theta \cos \psi} \\
\end{pmatrix} \tag{7}
\]
Parameters $a$, $c$ and $\mu$ determine the type of the cyclide. When $c < \mu \leq a$ it is a ring cyclide, when $0 < \mu \leq c$ it is a horned cyclide, and when $\mu > a$ it is a spindle cyclide.

A Dupin cyclide admits two planes of symmetry $\mathcal{P}_y : y = 0$ and $\mathcal{P}_z : z = 0$ which define two couples of circles, called principal circles, figures 2(a) and 2(b). From the knowledge of a couple of principal circles and the Dupin cyclide type, it is easy to calculate Dupin cyclide parameters [Gar07].

Curvatures lines of a Dupin cyclide are circles obtained with $\theta$ or $\psi$ constant in equation (7), figure 3.

It is known that there is a third family of circles onto a ring Dupin cyclide: the Villarceau circles [Gar08]. Then, one can construct, onto a Dupin cyclide, 3D triangles with circular edges, figure 4.

The planes containing circles of curvature of a Dupin cyclide form two pencils of planes, figure 5, and define two straight lines $\Delta_\theta$ as the intersection of the planes of the first pencil and $\Delta_\psi$ as the intersection of the planes of the other pencil. If the Dupin cyclide is a torus, the line $\Delta_\psi$ belongs to the infinite plane (the planes containing circles of curvature are parallel).

Several authors have proposed algorithms to convert a Dupin cyclide patch into a Bézier surface [Pra90, Ued95, AD96, FGP05, Gar07] and vice-versa [Gar07, GFN06].

Table 1 gives the four most important properties of control points of a Bézier surface obtained by the conversion of a Dupin cyclide patch, figure 7.

In table 1, property (PG4) can be presented as:

$$ P_{11} \in \text{Aff} \{ P_{00}; P_{01}; P_{10} \} \cap \text{Aff} \{ P_{02}; P_{12}; P_{01} \} \cap \text{Aff} \{ P_{20}; P_{21}; P_{10} \} \cap \text{Aff} \{ P_{22}; P_{21}; P_{12} \} $$  

where $\text{Aff} \{ A; B; C \}$ designate the affine space generated by points $A$, $B$ and $C$.
3 Constructions of 3D triangles

3.1 Aim

In this section we present and discuss an algorithm for the computation of a 3D triangle edged by three arc of circles. Two edges of this triangle are arc of circles of curvature, precisely a meridian and a parallel, the third edge is an arc of Villarceau circle. First of all, we will build the circular edges of the triangle and then we will construct a rational biquadratic Bézier surface and use a conversion algorithm [Gar07, GFN06] to get the parameters of the corresponding triangular Dupin cyclide patch (the 3D triangle).

3.2 Construction of the triangle

**Algorithm 1**: Construction of a 3D triangle with circular edges.

**Input**: Three non-colinear points $P_{00}, P_{02}$ and $P_{20}$ belonging to the plane of equation $z = 0$.

1. Choice of $P_{01}$ in the perpendicular bisector plane of $[P_{00}, P_{02}]$.
2. Construction of $P_{10}$ using (PG2) and (PG3) properties.
3. Choice of $P_{21}$ using (PG1) property.
4. Construction of $P_{12}$ et $P_{21}$ using (PG2), (PG3) and the conditions of perspectivity [ABG09].
5. Construction of $P_{11}$ using (PG4) property.
6. Computation of weights $\omega_{00}, \omega_{02}, \omega_{20}$ and $\omega_{22}$ using property (PG1).
7. Computation of weights $\omega_{10}$ et $\omega_{12}$ using theorem 1 such that $\omega_{10}\omega_{12} > 0$.
8. Computation of weights $\omega_{01}$ et $\omega_{21}$ using theorem 1 with the condition $\omega_{01}\omega_{21} > 0$.
9. Computation of weight $\omega_{11}$ using theorem 1 such that the median Bézier curves are circular arcs.
10. Points $P_{ij}(x_{ij}, y_{ij}, z_{ij}, w_{ij}), i = 0, 1, 2, j = 0, 1, 2$ constitute the control polyhedra of a biquadratic Bézier surface. This surface is converted into a Dupin cyclide patch [Gar07].
11. Construction of a Villarceau circle belonging to the Dupin cyclide patch.
12. Computation of the intersection points between the Villarceau circle and each circular arc of curvature.
13. Determination of the vertices of the triangle.

**Output**: A triangular Dupin cyclide patch with circular edges.

We start with three non-colinear points, $P_{00}, P_{02}$ and $P_{20}$. Without loss of generality, we consider these three points in the plane of equation $z = 0$. In order to have a circular arc, see figure 8, we choose $P_{01}$ to belong to the per-
pendicular bissector plane of \([P_{00}, P_{02}]\) (see property (PG2) in table 1).

During the second step, figure 8, we compute \(P_{10}\) with the help of properties (PG2) and (PG3):

- \(P_{10}\) belongs to the perpendicular bissector plane of the segment \([P_{00}, P_{02}]\)
- \(P_{00}P_{01} \perp P_{00}P_{10}\)
- \(P_{10}\) and \(P_{01}\) are in the same half-space:

\[ Z^+ = \{ M(x, y, z) \in \mathcal{E}; z > 0 \} \]

We can note that we have one degree of freedom because we solve a system with two equations and three variables.

During the third step, we use property (PG1) such that \(P_{22}\) belongs to \(\mathcal{C}\). Let \(O(x_0, y_0, 0)\) the center and \(R\) the radius of \(\mathcal{C}\). The points \(P_{00}, P_{02}, P_{20}, P_{22}\) can be defined by:

\[
\begin{align*}
P_{00} &= (x_0 + R \cos(\theta_{00}), y_0 + R \sin(\theta_{00}), 0) \\
P_{02} &= (x_0 + R \cos(\theta_{02}), y_0 + R \sin(\theta_{02}), 0) \\
P_{20} &= (x_0 + R \cos(\theta_{20}), y_0 + R \sin(\theta_{20}), 0) \\
P_{22} &= (x_0 + R \cos(\theta_{22}), y_0 + R \sin(\theta_{22}), 0)
\end{align*}
\]

Of course, in order to avoid obtaining a cross quadrangle, it is enough to fix a value of \(\theta_{22}\) such that \(P_{22}\) does not belong to the arc of circle defined by \(P_{02}, P_{20}\) and \(P_{00}\).

During the fourth step, figure 9, we will build the points \(P_{12}\) et \(P_{21}\) with the help of properties (PG2) and (PG3):

- \(P_{12}\) belongs to the perpendicular bissector plane of the segment \([P_{02}, P_{22}]\)
- \(P_{21}\) belongs to the perpendicular bissector plane of the segment \([P_{20}, P_{22}]\)
- \(P_{02}P_{01} \perp P_{02}P_{12}\)
- \(P_{20}P_{10} \perp P_{20}P_{21}\)
- \(P_{12}P_{22} \perp P_{21}P_{22}\)

As we have to solve a system with five equations and six variables, the last variable is determined by the perspectivity conditions [ABG09].

During the fifth step, we can easily compute the point \(P_{11}\) with the help of property (PG4).

During step six, we use property (PG1) in order to compute the weights \(\omega_{00}, \omega_{02}, \omega_{20}\) et \(\omega_{22}\).

During step seven, we use theorem 1 in order to compute the weights \(\omega_{10}, \omega_{12}\) with \(\omega_{10}\omega_{12} > 0\), and the weights \(\omega_{21}\) and \(\omega_{01}\) with \(\omega_{01}\omega_{21} > 0\) [Gar07], figure 10.

Then using theorem 2 and fixing \(G^v_i\) in the perpendicular bissector plane of \([G^v_0, G^v_2]\) and fixing \(G^v_i\) in the perpendicular bissector plane of \([G^v_0, G^v_2]\), we can compute the weight \(\omega_{11}\) during step nine. Points \(P_{ij}(x_{ij}, y_{ij}, z_{ij}, w_{ij}), i = 0, 1, 2, j = 0, 1, 2\) constitute the control polyhedra of a bi-quadratic Bézier surface, figure 11.

During step ten, we use a conversion algorithm [Gar07, GFN06] in order to obtain the Dupin cyclide corresponding to the Bézier surface defined by the level headed control points \(P_{i,j}, i, j \in [0, 2]\), figure 12.

During step eleven, we compute one of the Villarceau circles which passes through the Dupin cyclide we found at step ten.
During step twelve, we compute the intersection of the Villarceau circle and the two arcs of curvature belonging to the Dupin cyclide patch.

During the last step we compute the vertices $Q_{02}$ and $Q_{20}$. Let’s recall that $P_{00}$ is a vertex of the triangle. Then if the Villarceau circle passes through $P_{02}$ or $P_{20}$, then these ones are vertices. In the last case, the vertices are the points computed in step twelve.

Finally, using the algorithm 1, we can compute a 3D triangle, with circular edges, onto a Dupin cyclide. Figure 13 shows two adjacent triangles forming a rectangular patch on the Dupin cyclide. The figure show also the three families of circles (meridian, parallel, Villarceau circle) onto the Dupin cyclide. The blue and red circles (meridians and parallels) are lines of curvature. The black circle (the one which divide the rectangle in two adjacent triangles) is the Villarceau circle.

### 4 Conclusion

This work presents an algorithm to construct a 3D triangle with circular edges belonging to a Dupin cyclide patch. One of the drawbacks of this algorithm is that we can not choose the three vertices of the 3D triangle. So, an important improvement of this work will be to allow easy selection of these points.

Another interesting extension of this work is to construct a triangle with two arcs of Villarceau circle and one parallel circle arc or one meridian circle arc. We can also study the geometric conditions in order to make a $G^1$-join between two 3D triangles. Another possible extension of this work could be a construction algorithm of triangular Bézier surfaces belonging to triangular Dupin cyclide patches, and a conversion algorithm.

### References


Figure 13: Two 3D triangles with circular edges computed, on a Dupin cyclide, using algorithm 1.


