An algorithm to construct 3D triangles with circular edges

Bertrand Belbis[1], Lionel Garnier[1], Sebti Foufou[1,2]
(1) LE2I Lab, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France
bbelbis@u-bourgogne.fr, lgarnier@u-bourgogne.fr
(2) CSE Dept., College of Engineering, Qatar University, P.O. Box 2713, Doha Qatar
sfoufou@u-bourgogne.fr, sfoufou@qu.edu.qa

ABSTRACT

Dupin cyclides are non-spherical algebraic surfaces of degree 4, discovered by the French mathematician Pierre-Charles Dupin at the beginning of the 19th century. A Dupin cyclide has a parametric equation and two implicit equations and circular lines of curvature. It can be defined as the image of a torus, a cone of revolution or a cylinder of revolution by an inversion. A torus has two families of circles : meridians and parallels. There is a third family of circles on a ring torus: Villarceau circles. As the image, by an inversion, of a circle is a circle or a straight line, there are three families of circles onto a Dupin cyclide too. The goal of this paper is to construct, onto a Dupin cyclide, 3D triangles with circular edges: a meridian arc, a parallel arc and a Villarceau circle arc.
Keywords: *3D Triangles, circular edges, Dupin cyclides, rational biquadratic Bézier surfaces.*

1 Introduction

In C.A.D, a complex object can be represented by a triangular mesh. This representation gives a lot of freedom for the design, but it needs a large number of topological conditions. The storage and the visualization of complex meshes may be very costly in memory and time, fundamental geometric operations such as intersections may also be very heavy when they are carried on a several complex meshes in the same time.

It would be interesting to have an object formed by 3D triangles witch have an algebraic and a parametrical representation. On each triangle no topological information is needed. The parametric and implicit equations of the 3D triangles will be used for shape modeling, and to generate some planar approximations for the visualization purposes. In this work, we propose to computed 3D triangle on a family of algebraic surfaces, called Dupin cyclides. Dupin cyclides are degree 4 algebraic surfaces discovered by the French mathematician Pierre-Charles Dupin in 1822. These surfaces have a number of properties (e.g. circular lines of curvature, principal circles, ...) that facilitate their use in geometric modeling. The property that lists the families of circles that can be drawn on a Dupin cyclide is of the most importance to the work presented in this paper. Dupin cyclides have three families of circles : the parallels, the meridians and the Villarceau ones. The aim of this work is to build 3D triangles belonging to a Dupin cyclide, where the edges are arcs of parallel, meridian and Villarceau circles.

The paper is organized as follow: in section 2, we recall some definitions and properties of rational quadratic Bézier curves, definitions of rational biquadratic Bézier surfaces and definition and properties of Dupin cyclides. Section 3 presents the proposed algorithm for the construction of 3D triangles on a Dupin cyclide. Section 4 concludes the paper.

2 Background

2.1 Rational Bézier curves and surfaces

Rational quadratic Bézier curves are second degree parametric curves defined by:

$$\overrightarrow{OM(t)} = \frac{\sum_{i=0}^{2} w_i B_i(t) \overrightarrow{OP_i}}{\sum_{i=0}^{2} w_i B_i(t)},\ t \in [0,1] \qquad (1)$$

where $B_i(t)$ are quadratic Bernstein polynomials defined as:

$$B_0(t) = (1-t)^2\,, B_1(t) = 2t\,(1-t) \text{ and } B_2(t) = t^2$$

and for $i \in \{0,1,2\}$, w_i are weights associated with the control points P_i. For a standard rational quadratic Bézier curve, w_0 and w_2 are equal to 1, while w_1 can be used to control the type of conic defined by the curve [Far93, Far99, Gar07].

As we will modelize circle arcs using rational quadratic Bézier curve in our algorithm, we recall a theorem to compute the weight ω_1:

Theorem 1 *Circle defined by two points and its tangents at this points*

Let P_0, P_1 and P_2 three non-colinear points. (P_0P_1) and (P_2P_1) are the tangents to the circle $\mathcal{C}$ at P_0 and P_2. The center of $\mathcal{C}$ is O_0 and the radius of $\mathcal{C}$ is R.

Let I_1 be the middle of $[P_0, P_2]$.

Let $\mathcal{P}$ the perpendicular bissector plane of $[P_0, P_2]$.

Let $\mathcal{P}_c$ be the plane containing the Bézier curve. Let $(\omega_0, \omega_2) \in (\mathbb{R}_*^+)^2$ and $\omega = \omega_0 + \omega_2$.

Let $G = bar\{(P_0, \omega_0), (P_2, \omega_2)\}$ (bar is a notation for barycentre).

The rational quadratic Bézier curve γ of level-headed control points $(P_0, \omega_0), (P_1, \omega_1)$ and (P_2, ω_2) is an arc of circle if and only if:

$$\left\| \overrightarrow{O_0\gamma\left(\frac{1}{2}\right)} \right\| = R \tag{2}$$

which is equivalent to the equation $\alpha\omega_1^2 + \beta = 0$, where α and β are given by:

$$\alpha = 4(R^2 - O_0P_1^2), \quad \beta = \omega^2(R^2 - O_0G^2)$$

The positive solution ω_1^+ (resp. negative ω_1^+) of the solutions

$$\omega_1^+ = \sqrt{\frac{-\beta}{\alpha}}, \quad \omega_1^- = -\sqrt{\frac{-\beta}{\alpha}} \tag{3}$$

let us design the small (resp. the great) circle arc of the circle $\mathcal{C}$ with P_0 and P_2 as bounds.

Let us note that if $\omega_0 = \omega_2 = 1$ then the computation of weight ω_1 is simplified to:

$$\omega_1^+ = \cos\left(\widehat{\overrightarrow{P_0P_1}, \overrightarrow{P_0P_1}}\right), \quad \omega_1^- = -\cos\left(\widehat{\overrightarrow{P_0P_1}, \overrightarrow{P_0P_1}}\right)$$

Rational biquadratic Bézier surfaces are defined by control points $(P_{ij})_{0 \leq i,j \leq 2}$ and weights $(w_{ij})_{0 \leq i,j \leq 2}$ as:

$$\overrightarrow{OM(u,v)} = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j(v) \overrightarrow{OP_{ij}}}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j(v)} \tag{4}$$

The following theorem for the computation of the barycentric middle curve on a Bézier surface will be used in the Section 3. The proof of this theorem can be found in [GBF09]:

Theorem 2 *Barycentric middle curve*

Let us consider a Bézier surface defined by control points $(P_{ij})_{0 \leq i,j \leq 2}$ and weights $(w_{ij})_{0 \leq i,j \leq 2}$.

Let $G_i^u = \mathsf{bar}\{(P_{i0}; w_{i0}), (P_{i1}; 2w_{i1}), (P_{i2}; w_{i2})\}$ and $\alpha_i^u = w_{i0} + 2w_{i1} + w_{i2}$ where $i \in [\![0;2]\!]$.

Let $G_i^v = \mathsf{bar}\{(P_{0i}; w_{0i}), (P_{1i}; 2w_{1i}), (P_{2i}; w_{2i})\}$ and $\alpha_i^v = w_{0i} + 2w_{1i} + w_{2i}$ where $i \in [\![0;2]\!]$.

If $\sum_{i=0}^{2} \alpha_i^u \neq 0$, the barycentric middle curve

$$u \mapsto M\left(u, \frac{1}{2}\right)$$

is a Bézier curve with control points $(G_i^u; \alpha_i^u)_{i \in [\![0;2]\!]}$.

If $\sum_{i=0}^{2} \alpha_i^v \neq 0$, the barycentric middle curve

$$v \mapsto M\left(\frac{1}{2}, v\right)$$

is a Bézier curve with control points $(G_i^v; \alpha_i^v)_{i \in [\![0;2]\!]}$.

More details on Bézier surfaces can be found in [Far93, Far99, Gar07]. In the remaining of this paper, we only consider rational Bézier curves and surfaces of degree two to which we refer, for short, by Bézier curves and Bézier surfaces.

2.2 Dupin cyclides

Non-degenerate Dupin cyclides, figure 1, have been definied by P. Dupin [Dup22]. A simple definition is: a Dupin cyclide is the image of a torus, a circular cone or a circular cylinder by an inversion [Gar07].

A. R. Forsyth [For12] and G. Darboux [For12, Dar17] have given two equivalent implicit equations:

$$\left(x^2 + y^2 + z^2 - \mu^2 + b^2\right)^2 = 4\left(ax - c\mu\right)^2 + 4b^2y^2 \tag{5}$$

$$\left(x^2 + y^2 + z^2 - \mu^2 - b^2\right)^2 = 4\left(cx - a\mu\right)^2 - 4b^2z^2 \tag{6}$$

in an orthonormal basis $(O, \overrightarrow{\imath_0}, \overrightarrow{\jmath_0}, \overrightarrow{k_0})$, where O is called Dupin cyclide center, and parameters a, b and c are related by $c^2 = a^2 - b^2$. The parameter a is always greater than or equal to c.

A parametric equation of a Dupin cyclide is:

$$\Gamma_d(\theta, \psi) = \begin{pmatrix} \dfrac{\mu(c - a\cos\theta\cos\psi) + b^2\cos\theta}{a - c\cos\theta\cos\psi} \\ \dfrac{bsin\theta \times (a - \mu\cos\psi)}{a - c\cos\theta\cos\psi} \\ \dfrac{b\sin\psi \times (c\cos\theta - \mu)}{a - c\cos\theta\cos\psi} \end{pmatrix} \tag{7}$$

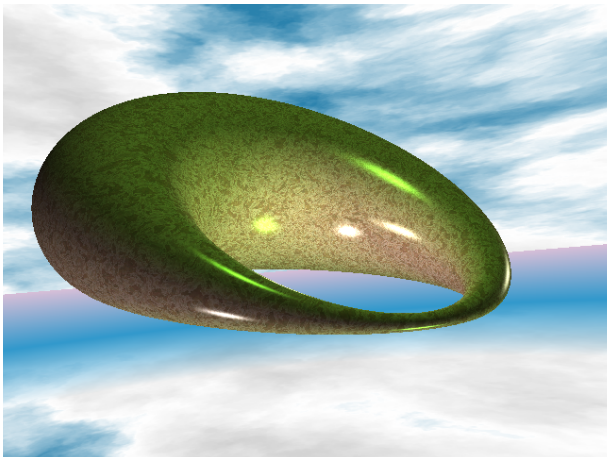
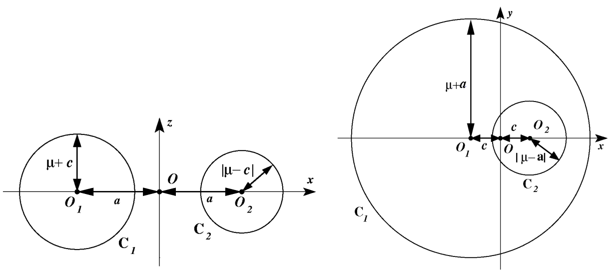


Figure 1: A ring Dupin cyclide.

Parameters a, c and μ determine the type of the cyclide. When $c < \mu \leq a$ it is a ring cyclide, when $0 < \mu \leq c$ it is a horned cyclide, and when $\mu > a$ it is a spindle cyclide.

A Dupin cyclide admits two planes of symmetry $\mathcal{P}_y : y = 0$ and $\mathcal{P}_z : z = 0$ which define two couples of circles, called principal circles, figures 2(a) and 2(b). From the knowledge of a couple of principal circles and the Dupin cyclide type, it is easy to calculate Dupin cyclide parameters [Gar07].



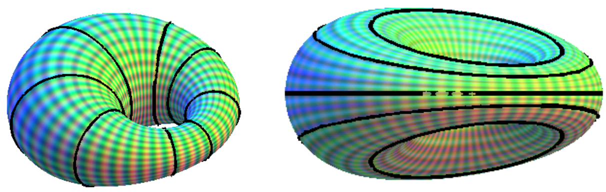
(a) (b)

Figure 2: Principal circles of ring Dupin cyclides. (a) : in plane $\mathcal{P}_y : y = 0$. (b) : in plane $\mathcal{P}_z : z = 0$.

Curvatures lines of a Dupin cyclide are circles obtained with θ or ψ constant in equation (7), figure 3.

It is known that there is a third family of circles onto a ring Dupin cyclide : the Villarceau circles [Gar08]. Then, one can construct, onto a Dupin cyclide, 3D triangles with circular edges, figure 4.

The planes containing circles of curvature of a Dupin cyclide form two pencils of planes, figure 5, and define two straight lines Δ_θ as the intersection of the planes of the first pencil and Δ_ψ as the intersection of the planes of the other pencil. If the Dupin cyclide is a torus, the line Δ_ψ belongs to the infinity plane (the planes containing circles of curvature are parallel).



(a) (b)

Figure 3: Dupin cyclide curvature lines are circles: (a) : meridians, θ is constant. (b) : parallels, ψ is constant.

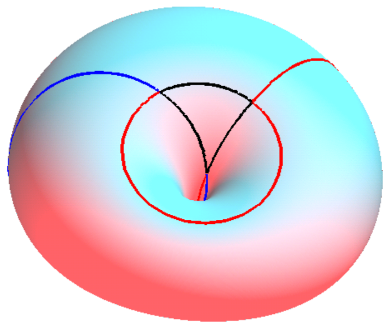
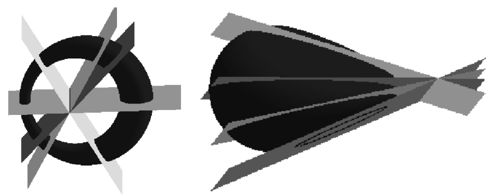


Figure 4: A 3D Triangle onto a ring Dupin cyclide.



(a) (b)

Figure 5: Two pencils of planes generated by Dupin cyclide curvature circles defining Δ_θ (a) and Δ_ψ (b).

Figure 6 shows lines Δ_θ and Δ_ψ with the principal circles of the ring Dupin cyclide in plane $\mathcal{P}_y$ (C_1^θ, C_2^θ) and in plane $\mathcal{P}_z$ (C_1^ψ, C_2^ψ). Δ_0 is the common perpendicular to Δ_θ and Δ_ψ. More details about properties of Dupin cyclide can be found in [Pra90, She98, AD96, DMP93].

Several authors have proposed algorithms to convert a Dupin cyclide patch into a Bézier surface [Pra90, Ued95, AD96, FGP05, Gar07] and vice-versa [Gar07, GFN06].

Table 1 gives the four most important properties of control points of a Bézier surface obtained by the conversion of a Dupin cyclide patch, figure 7.

In table 1, property (PG4) can be presented as:

$$\begin{aligned} P_{11} \in Aff\,\{P_{00}; P_{01}; P_{10}\} \cap Aff\,\{P_{02}; P_{12}; P_{01}\} \\ \cap Aff\,\{P_{20}; P_{21}; P_{10}\} \cap Aff\,\{P_{22}; P_{21}; P_{12}\} \end{aligned} \tag{8}$$

where $Aff\,\{A; B; C\}$ designate the affine space generated by points A, B and C.

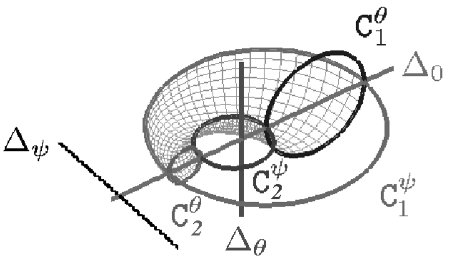


Figure 6: Straight lines Δ_θ and Δ_ψ obtained as intersections of two pencil planes.

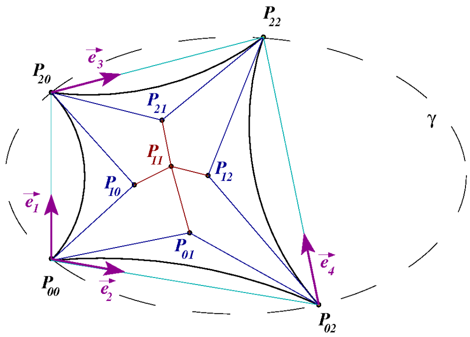


Figure 7: Level-headed control points schema of a Bezier surface obtained by a Dupin cyclide conversion.

(PG1)	P_{00}, P_{02} , P_{22} et P_{20} are cocyclical
(PG2)	$P_{00}P_{01} = P_{01}P_{02} \quad P_{02}P_{12} = P_{12}P_{22}$ $P_{22}P_{21} = P_{21}P_{20} \quad P_{00}P_{10} = P_{10}P_{20}$
(PG3)	$\overrightarrow{P_{00}P_{10}} \perp \overrightarrow{P_{00}P_{01}} \quad \overrightarrow{P_{02}P_{01}} \perp \overrightarrow{P_{02}P_{12}}$ $\overrightarrow{P_{22}P_{12}} \perp \overrightarrow{P_{22}P_{21}} \quad \overrightarrow{P_{20}P_{21}} \perp \overrightarrow{P_{20}P_{10}}$
(PG4)	$\overrightarrow{P_{00}P_{11}} \bullet \left(\overrightarrow{P_{00}P_{10}} \times \overrightarrow{P_{00}P_{01}}\right) = 0$ $\overrightarrow{P_{02}P_{11}} \bullet \left(\overrightarrow{P_{02}P_{01}} \times \overrightarrow{P_{02}P_{12}}\right) = 0$ $\overrightarrow{P_{22}P_{11}} \bullet \left(\overrightarrow{P_{22}P_{12}} \times \overrightarrow{P_{22}P_{21}}\right) = 0$ $\overrightarrow{P_{20}P_{11}} \bullet \left(\overrightarrow{P_{20}P_{21}} \times \overrightarrow{P_{20}P_{10}}\right) = 0$

Table 1: Geometrical properties of a Bézier surface obtained by conversion of a Dupin cyclide patch.

3 Constructions of 3D triangles

3.1 Aim

In this section we present and discuss an algorithm for the computation of a 3D triangle edged by three arc of circles. Two edges of this triangle are arc of circles of curvature, precisely a meridian and a parallel, the third edge is an arc of Villarceau circle. First of all, we will build the circular edges of the triangle and then we will construct a rational biquadratic Bézier surface and use a conversion algorithm [Gar07, GFN06] to get the parameters of the corresponding triangular Dupin cyclide patch (the 3D triangle).

3.2 Construction of the triangle

Algorithm 1 : Construction of a 3D triangle with circular edges.

Input: Three non-colinear points P_{00}, P_{02} and P_{20} belonging to the plane of equation $z = 0$.

1. Choice of P_{01} in the perpendicular bissector plane of $[P_{00}, P_{02}]$.
2. Construction of P_{10} using (PG2) and (PG3) properties.
3. Choice of P_{22} using (PG1) property.
4. Construction of P_{12} et P_{21} using (PG2), (PG3) and the conditions of perspectivity [ABG09].
5. Construction of P_{11} using (PG4) property.
6. Computation of weights ω_{00}, ω_{02}, ω_{20} and ω_{22} using property (PG1).
7. Computation of weights ω_{10} et ω_{12} using theorem 1 with the condition $\omega_{10}\omega_{12} > 0$.
8. Computation of weights ω_{01} and ω_{21} using theorem 1 with the condition $\omega_{01}\omega_{21} > 0$.
9. Computation of weight ω_{11} using theorem 1 such that the median Bézier curves are circular arcs.
10. Points $P_{ij}(x_{ij}, y_{ij}, z_{ij}, w_{ij}), i = 0, 1, 2, j = 0, 1, 2$ constitute the control polyhedra of a biquadratic Bézier surface. This surface is converted into a Dupin cyclide patch [Gar07].
11. Construction of a Villarceau circle belonging to the Dupin cyclide patch.
12. Computation of the intersection points between the Villarceau circle and each circular arc of curvature.
13. Determination of the vertices of the triangle.

Output: A triangular Dupin cyclide patch with circular edges.

We start with three non-colinear points, P_{00}, P_{02} and P_{20}. Without loss of generality, we consider these three points in the plane of equation $z = 0$. In order to have a circular arc, see figure 8, we choose P_{01} to belong to the perpendicular bisector plane of $[P_{00}, P_{02}]$ (see property (PG2) in table 1).

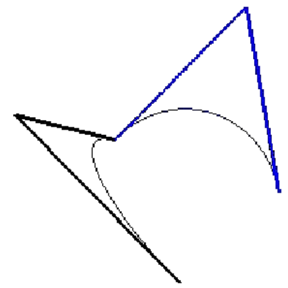


Figure 8: First and second steps of the algorithm 1.

During the second step, figure 8, we compute P_{10} with the help of properties properties (PG2) and (PG3):

- P_{10} belongs to the perpendicular bisector plane of the segment $[P_{00}P_{20}]$
- $\overrightarrow{P_{00}P_{01}} \perp \overrightarrow{P_{00}P_{10}}$
- P_{10} and P_{01} are in the same half-space:

$$\mathcal{Z}^+ = \{M(x, y, z) \in \mathcal{E}; z > 0\}$$

We can note that we have one degree of freedom because we solve a system with two equations and three variables.

During the third step, we use property (PG1) such that P_{22} belongs to $\mathcal{C}$. Let $O(x_o, y_o, 0)$ the center and R the radius of $\mathcal{C}$. The points P_{00}, P_{02}, P_{20}, P_{22} can be defined by:

$$\begin{array}{l} P_{00} = (x_O + R\cos(\theta_{00}), y_O + R\sin(\theta_{00}), 0) \\ P_{02} = (x_O + R\cos(\theta_{02}), y_O + R\sin(\theta_{02}), 0) \\ P_{20} = (x_O + R\cos(\theta_{20}), y_O + R\sin(\theta_{20}), 0) \\ P_{22} = (x_O + R\cos(\theta_{22}), y_O + R\sin(\theta_{22}), 0) \end{array}$$

Of course, in order to avoid obtaining a cross quadrangle, it is enough to fix a value of θ_{22} such that P_{22} does not belong to the arc of circle defined by P_{02}, P_{20} and P_{00}.

During the fourth step, figure 9, we will build the points P_{12} et P_{21} with the help of properties (PG2) and (PG3):

- P_{12} belongs to the perpendicular bisector plane of the segment $[P_{02}P_{22}]$
- P_{21} belongs to the perpendicular bisector plane of the segment $[P_{20}P_{22}]$
- $\overrightarrow{P_{02}P_{01}} \perp \overrightarrow{P_{02}P_{12}}$
- $\overrightarrow{P_{20}P_{10}} \perp \overrightarrow{P_{20}P_{21}}$
- $\overrightarrow{P_{12}P_{22}} \perp \overrightarrow{P_{21}P_{22}}$

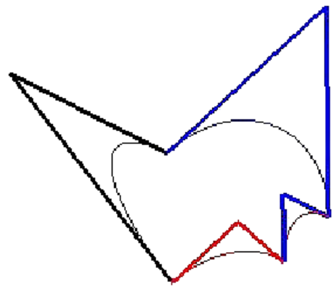


Figure 9: Fourth step of the algorithm 1.

As we have to solve a system with five equations and six variables, the last variable is determined by the perspectivity conditions [ABG09].

During the fifth step, we can easily compute the point P_{11} with the help of property (PG4).

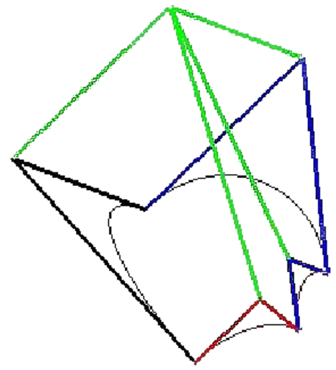


Figure 10: Steps 5, 6, 7 and 8 of the algorithm 1.

During step six, we use property (PG1) in order to compute the weights ω_{00}, ω_{02}, ω_{20} et ω_{22}.

During step seven, we use theorem 1 in order to compute the weights ω_{10}, ω_{12} with $\omega_{10}\omega_{12} > 0$, and the weights ω_{21} and ω_{01} with $\omega_{01}\omega_{21} > 0$ [Gar07], figure 10.

Then using theorem 2 and fixing G_1^u in the perpendicular bisector plane of $[G_0^u, G_2^u]$ and fixing G_1^v in the perpendicular bisector plane of $[G_0^v, G_2^v]$, we can compute the weight ω_{11} during step nine. Points $P_{ij}(x_{ij}, y_{ij}, z_{ij}, w_{ij}), i = 0, 1, 2, j = 0, 1, 2$ constitute the control polyhedra of a biquadratic Bézier surface, figure 11.

During step ten, we use a conversion algorithm [Gar07, GFN06] in order to obtain the Dupin cyclide corresponding to the Bézier surface defined by the level headed control points $P_{i,j}$,$i, j \in [0, 2]$, figure 12.

During step eleven, we compute one of the Villarceau circles which passes through the Dupin cyclide we found at step ten.

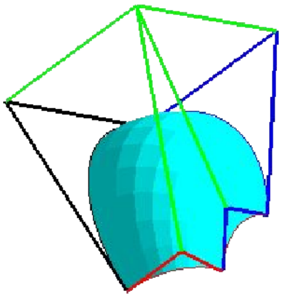


Figure 11: Bézier surface, step 9 of algorithm 1.

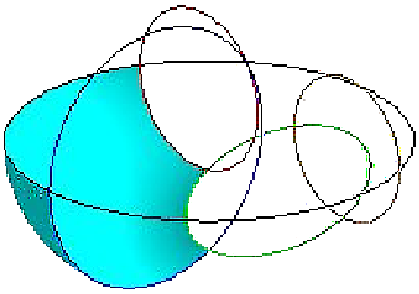


Figure 12: Steps 10 of the algorithm 1.

During step twelve, we compute the intersection of the Villarceau circle and the two arcs of curvature belonging to the Dupin cyclide patch.

During the last step we compute the verticies Q_{02} and Q_{20}. Let's recall that P_{00} is a vertex of the triangle. Then if the Villarceau circle passes through P_{02} or P_{20} then these ones are verticies. In the last case, the vertices are the points computed in step twelve.

Finally, using the algorithm 1, we can compute a 3D triangle, with circlar edges, onto a Dupin cyclide. Figure 13 shows two adjacent triangles forming a rectangular patch on the Dupin cyclide. The figure show also the three families of circles (meridian, parallel, Villarceau circle) onto the Dupin cyclide. The blue and red circles (meridians and parallels) are lines of curvature. The black circle (the one which divide the rectangle in two adjacent triangles) is the Villarceau circle.

4 Conclusion

This work presents an algorithm to construct a 3D triangle with circular edges belonging to a Dupin cyclide patch. One of the drawbacks of this algorithm is that we can not choose the three vertices of the 3D triangle. So, an important improvement of this work will be to allow easy selection of these points.

Another interesting extension of this work is to construct a triangle with two arcs of Villarceau circle and one parallel circle arc or one meridian circle arc. We can also study the geometric conditions in order to make a G^1-join between two 3D triangles. Another possible extension of this work could be a construction algorithm of triangular Bézier surfaces belonging to triangular Dupin cyclide patches, and a conversion algorithm.

References

[ABG09] G. Albrecht, B. Belbis, and L. Garnier. Modlisation d'une surface de Bézier rationnelle biquadratique convertible en un carreau de supercyclide. In *GTMG 2009, Toulouse*, pages 77–85, 2009.

[AD96] G. Albrecht and W. Degen. Construction of Bézier rectangles and triangles on the symetric Dupin horn cyclide by means of inversion. *Computer Aided Geometric Design*, 14(4):349–375, 1996.

[Dar17] G. Darboux. *Principes de géométrie analytique*. Gauthier-Villars, 1917.

[DMP93] D. Dutta, R. R. Martin, and M. J. Pratt. Cyclides in surface and solid modeling. *IEEE Computer Graphics and Applications*, 13(1):53–59, January 1993.

[Dup22] C. P. Dupin. *Application de Géométrie et de Méchanique à la Marine, aux Ponts et Chaussées, etc.* Bachelier, Paris, 1822.

[Far93] G. Farin. *Curves And Surfaces*. Academic Press, 3rd edition, 1993.

[Far99] G. Farin. *NURBS from Projective Geometry to Pratical Use*. A K Peters, Ltd, 2 edition, 1999. ISBN 1-56881-084-9.

[FGP05] S. Foufou, L. Garnier, and M. Pratt. Conversion of Dupin Cyclide Patches into Rational Biquadratic Bézier Form. In R. Martin, H. Bez, and M. Sabin, editors, *Proceedings of the 11th Conference on the Mathematics of Surfaces*, pages 201–218. Springer-Verlag Berlin Heidelberg, September 2005. ISBN: 3-540-28225.

[For12] A. R. Forsyth. *Lecture on Differential Geometry of Curves and Surfaces*. Cambridge University Press, 1912.

[Gar07] L. Garnier. *Mathématiques pour la modélisation géométrique, la représentation 3D et la synthèse d'images*. Ellipses, 2007.

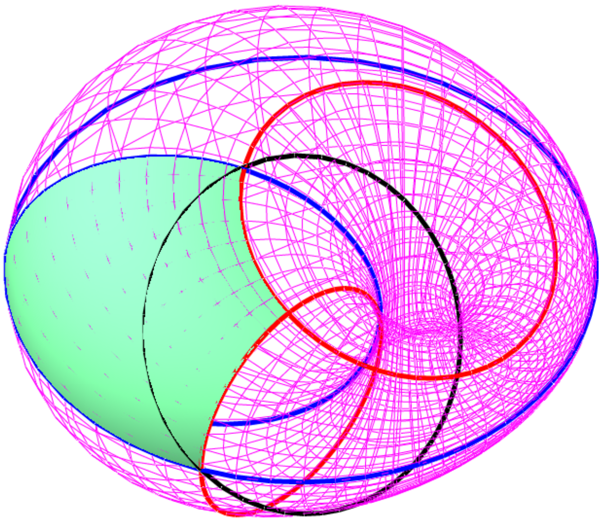


Figure 13: Two 3D triangles with circular edges computed, on a Dupin cyclide, using algorithm 1.

[Gar08] L. Garnier. Reprsentation analytique des cercles de Villarceau sur une cyclide de Dupin en anneau. *R.E.F.I.G.*, 2(1):47–59, 2008.

[GBF09] L. Garnier, B. Belbis, and S. Foufou. Conversion of biquadratic rational Bézier surfaces into patches of particular Dupin cyclides: the torus and the double sphere. In *WSCG09, Communication papers proceedings*, pages 101–107, University of West Bohemia, Plzen, Czech Republic, February 2009.

[GFN06] L. Garnier, S. Foufou, and M. Neveu. Conversion d'un carreau de Bézier rationnel biquadratique en un carreau de cyclide de Dupin quartique. *TSI*, 25(6):709–734, 2006. numéro spécial AFIG'04, Hermès.

[Pra90] M. J. Pratt. Cyclides in computer aided geometric design. *Computer Aided Geometric Design*, 7(1-4):221–242, 1990.

[She98] C. K. Shene. Blending two cones with Dupin cyclides. *Computer Aided Geometric Design*, 15(7):643–673, 1998.

[Ued95] K. Ueda. Normalized Cyclide Bézier Patches. In *Mathematical Methods for Curves and Surfaces*, pages 507–516, Nashville, USA, 1995. Vanderbilt University Press.