Noise induced breather generation in a sine-Gordon Chain

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Abstract. We consider a sine-Gordon chain sinusoidally driven at one end. In absence of noise, there exists a well known critical value of the amplitude beyond which breather modes can be generated via the phenomenon of supratransmission. We consider values of the driving amplitude below the critical amplitude such that no breather propagates in the medium. We show that noise induces breather generation with a given probability depending on the noise intensity. We also propose a bifurcation diagram which extends the supratransmission effect to a more realistic signal, namely a noisy sinusoidal excitation. We finally discuss some promising signal processing applications that can be developed by taking into account the contribution of noise in the media sharing this supratransmission phenomenon.

1. Introduction:

For more than ten years now, the discovery of Stochastic Resonance (SR) by Benzi in the context of climate dynamics has changed forever the way to consider noise in nonlinear media [1]. Indeed, under certain conditions, it has been shown that the response of a nonlinear system to a deterministic excitation can be enhanced by an appropriate amount of noise via S.R. [2]. This counter intuitive phenomenon has naturally encouraged researchers to include the contribution of noise in their investigations, leading to interesting applications in different fields such as modeling of biological systems [3, 4, 5, 6], signal processing [7, 8, 9, 10], image processing [11, 12, 13] or nonlinear information transmission [14, 15, 16, 17, 18], to cite but a few. In this last area, it has been demonstrated that if we drive a nonlinear pass-band electrical transmission line beyond its high cutoff frequency with a periodic excitation, the addition of an appropriate amount of noise would trigger soliton generation in the medium with a given probability [19]. In the same way but in the absence of noise, Geniet and Leon [20, 21] have theoretically studied a completely different nonlinear system. Indeed, considering a medium described by a sine-Gordon equation subjected to irradiation at a frequency in the stop gap, they have showed that, when the driving amplitude at the input boundary exceeds a threshold value, a large amount of energy flows through the medium by means of nonlinear modes generation. This phenomenon, known under the name of Nonlinear Supratransmission, has been reported in various nonlinear waveguides [22, 23, 24, 25], but most often in the deterministic case. Contrary to the works presented in reference [19], we choose here a driving frequency in the stop gap and we consider media described by the sine-Gordon equation. In particular, we investigate if the addition of noise can induce the nonlinear supratransmission effect in a range of parameters where it does not occur. Moreover, we mainly focus our study on breather modes generation. Indeed, breather modes have been shown to play a main part in storage and transport of energy [26, 27, 28]. More recently, it has been reported that they can be used to develop some novel applications in information transmission [24]. Therefore, the study of the generation of this nonlinear modes under the presence of noise is undoubtedly of great interest.

In the second section, we briefly introduce the sine-Gordon model and the supratransmission effect in the deterministic case. Especially, after a recall of the theoretical results established by Leon and Geniet [20, 21], we detail the numerical procedure used to simulate the sine-Gordon model. Then, in the third section, we investigate if an additive white noise can induce the supratransmission phenomenon and we provide a bifurcation diagram which extends the supratransmission effect to a noisy sinusoidal signal. In the last section, we finally suggest some outlooks and we discuss some applications that can be developed by considering noise in the sine-Gordon chain.

2. Discrete sine-Gordon Chain with a deterministic excitation

In this paper, we consider a discrete sine-Gordon chain of coupled oscillators $u_n(t)$ ruled by the following differential equation

$$\frac{d^2U_n}{dt^2} - c^2(U_{n+1} + U_{n-1} - 2U_n) + \omega_0^2 \sin U_n + \gamma_n \dot{U}_n = 0. \tag{1}$$

To assume an absorbing end, the damping coefficient γ_n in eq. (1) is null for all cells of the chain except for the last few cells. The profile of the damping coefficient is the following:

$$\gamma_n = 1 + \tanh\left(\frac{2n - 2N + m}{2b}\right). \tag{2}$$

When one end of the chain is sinusoidally driven with a weak amplitude, the wave number k and the angular frequency Ω of the harmonic wave propagating in the sine-Gordon chain obey to

$$\omega^2 = \omega_0^2 + 2c^2(1 - \cos k). \tag{3}$$

This dispersion relation corresponds to a typical bandpass filter with a gap defined by ω_0 and a cut-off angular frequency $\omega_{max} = \sqrt{\omega_0^2 + 4c^2}$. We will restrict our study to the case of angular frequencies in the forbidden band gap, that is below the gap ω_0. In this case, the linear theory leads to an evanescent wave with the following profile [20]

$$u_n(t) = A \sin(\Omega t) \exp(-\lambda n). \tag{4}$$

The coefficient λ is deduced setting $k = i\lambda$ in the dispersion relation (3)

$$\lambda = \text{arccosh}\left(1 + \frac{1 - \Omega^2}{2c^2}\right) \tag{5}$$

In our numerical experiments, we have used the same boundary condition used by Geniet and Leon in [20]:

- One end of the chain is driven sinusoidally with an angular frequency Ω and an amplitude A.
- The chain is initially at rest.
- The initial velocities of all particles of the chain are those of an evanescent wave to avoid the shock wave generated by initial null velocities.

Therefore, these boundary conditions reduce to

$$u_0(t) = A \sin(\Omega t), \qquad u_n(0) = 0, \qquad \dot{u}_n(0) = A\Omega e^{-\lambda n}. \tag{6}$$

Supratransmission occurs when the amplitude of the sinusoidal driving exceeds a critical value A^* that depends on the signal frequency. More precisely, a great value of energy penetrates in the medium by mean of nonlinear modes generation. According to Geniet and Leon, this critical amplitude is linked to the excitation frequency by the theoretical expression

$$A^* = 4 \arctan\left[\frac{c}{\Omega} \text{arccosh}\left(1 + \frac{1 - \Omega^2}{2c^2}\right)\right]. \tag{7}$$

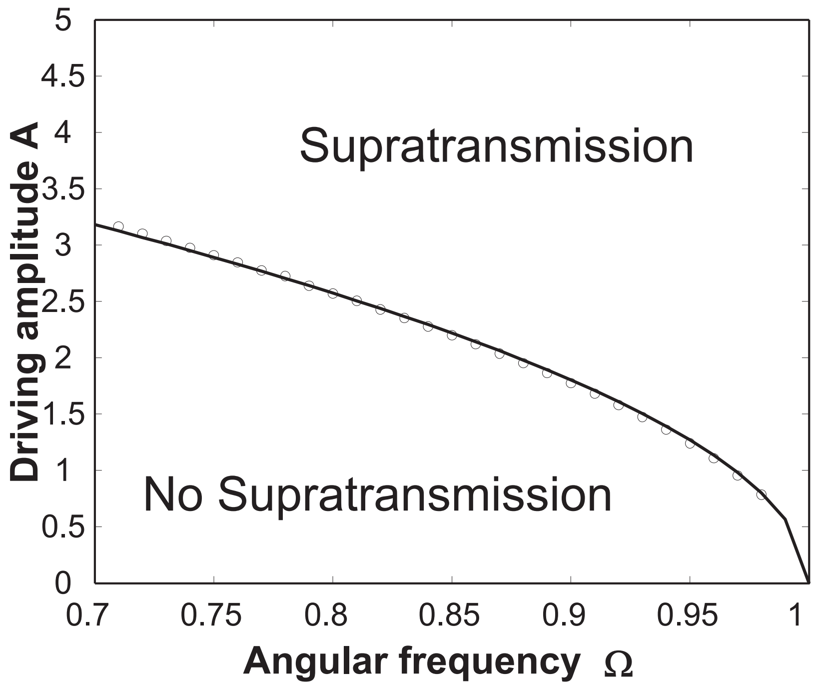


Figure 1. Critical amplitude A^* beyond which supratransmission occurs versus the angular frequency Ω. The theoretical law (7) is plotted in solid line while the (o) signs corresponds to numerical results. The parameters are $N = 4000$, $m = 500$, $b = 140$, $\omega_0 = 1$, $c = 10$.

We have first numerically investigated the supratransmission effect in the deterministic case since there exists a theoretical description of the phenomenon. Indeed, once the deterministic case validated, we will be able to investigate the effect of noise on this nonlinear phenomenon. To simulate the sine-Gordon chain ruled by eq. (1), we have used a standard fourth order Runge-Kutta algorithm with integrating time step $dt = 10^{-2}$ and with the boundary conditions (6). The critical amplitude A^* beyond which supratransmission occurs is determined by dichotomy after a direct integration of eq. (1).

In fig. 1, our numerical results match with a fairly good agreement the theoretical predictions, which allows to validate our numerical procedure in the deterministic case.

3. Sine-Gordon chain with a stochastic excitation

We now consider a sinusoidal excitation corrupted by an additive white gaussian noise $\eta(t)$ of Root Mean Square (R.M.S.) value σ. Moreover, as we focus our interest in breather modes generation, we replace for $t > t_0$ the boundary condition (6) by

$$u_0(t) = \exp\Big(-\frac{t - t_0}{(20\pi/\Omega)}\Big)\Big(A\sin(\Omega t) + \eta(t)\Big)H(t - t_0), \qquad (8)$$

where $H(t)$ is the Heaviside function and t_0 corresponds to the time for which the breather reaches the 50^{th} cell. With this modified boundary condition, the end of the medium is sinusoidally driven until only one breather propagates in the medium. Then, the amplitude of the driving is slowly decreased so that no other breathers are generated.

3.1. Numerical procedure

We have set the parameters of the chain to $\omega_0 = 1$, $c = 10$, while the angular frequency of the sinusoidal driving is adjusted to $\Omega = 0.95$. In absence of noise, the critical amplitude beyond which supratransmission occurs is then numerically obtained for $A^* = 1.24$. We consider an amplitude of excitation A below the critical value A^*, while the time of the simulation is set to $T = 2000$ and the size of the chain is $N = 4000$. The gaussian noise is numerically generated with the algorithm detailed in [29], while the differential equation (1) is solved with the standard fourth order runge-Kutta algorithm (RK-4) with integrating time step $dt = 10^{-2}$. Moreover, as the RK-4 methods requires samples every half integration time step, we estimate the noise at time step $t + dt/2$ by a linearized interpolation of the noise generated at time steps t and $t + dt$ [29].

In order to numerically investigate the noise effect on this supratransmission phenomenon, for each value of the RMS noise intensity, we have performed 200 runs starting the chain with the initial boundary conditions (6) and (8). Then, we analyze the probability of generating a breather mode versus the noise amplitude σ .

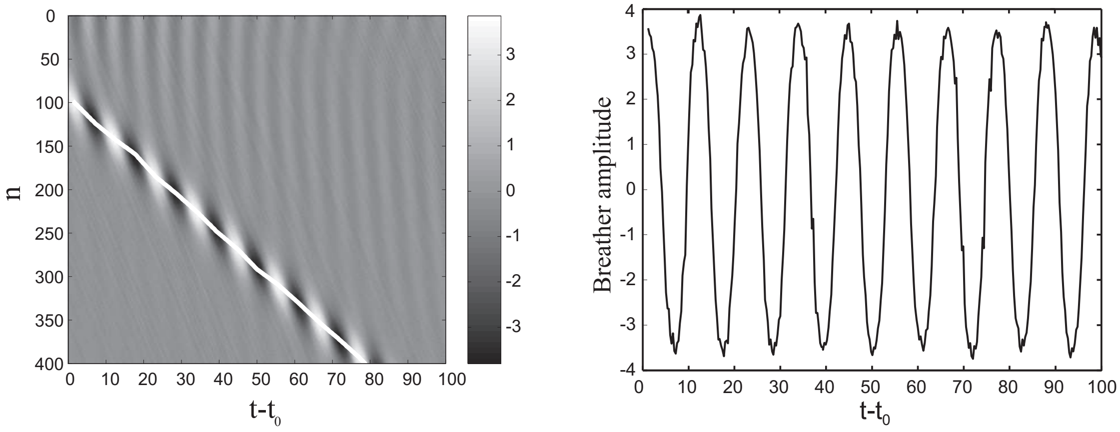
3.2. Numerical results

Our simulations have revealed that it is possible with an appropriate amount of additive noise to generate breather modes in the chain. The spatiotemporal diagram of Fig. 2.(a) shows the propagation of such breather in the specific case where the $R.M.S.$ noise amplitude value is $\sigma = 0.04$.

We have also presented the temporal evolution of the breather obtained tracking the breather along its propagative axis. This trajectory of the breather has been numerically determined by founding the local extrema in the spatiotemporal diagram of fig. 2. We have linearly interpolated the trajectory of the breather between two consecutive local extrema, which allows to track the breather along its propagative axis as shown by the white line of fig. 2.(a). It provides the temporal signal of fig. 2.(b) which exhibits a noisy sinusoidal behaviour, conveying that the breather is also corrupted by the additive white gaussian noise. To characterize the generation of breather induced by noise, we have estimated the probability to generate a breather over 200 simulations run.

The results are summarized in fig. 3 where the probability is plotted versus σ for different amplitudes of the excitation.

Each curve displays a staire case profile. Indeed, we observe that the probability first remains constant and equal to zero as σ is increased from zero up to some critical



(a) (b)

Figure 2. . (a) Spatiotemporal view of the breather generated in the sine-Gordon medium using a noisy sinusoidal excitation. The white line corresponds to the propagative axis of the breather. (b) Temporal evolution of the breather along its propagative axis. Parameters: $N = 4000$, $m = 500$, $b = 140$, $\omega_0 = 1$, $c = 10$, $\Omega = 0.95$, $A = 1.23$, $\sigma = 0.04$.

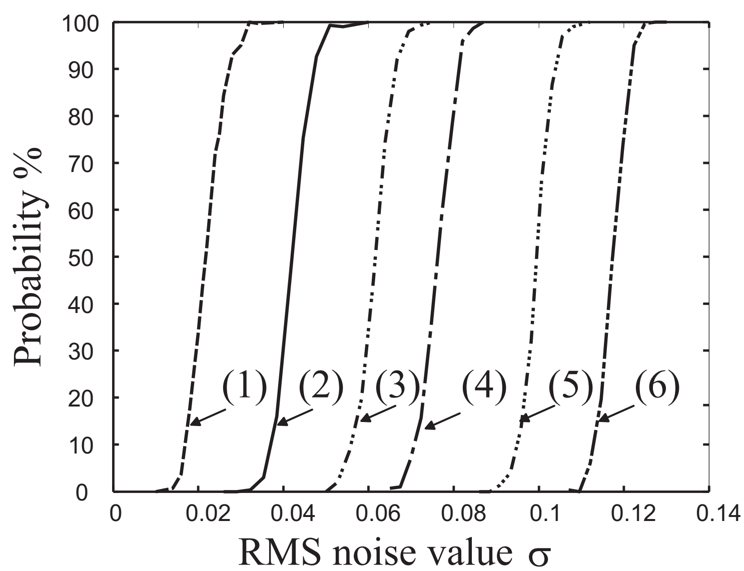


Figure 3. Probability to generate a breather versus the RMS noise amplitude σ for different amplitude of the sinusoidal driving. Parameters: $N = 4000$, $m = 500$, $b = 140$, $\omega_0 = 1$, $c = 10$, $\Omega = 0.95$. The probability is estimated over 200 simulations. (1) A=1.23, (2) A=1.2, (3) A=1.15, (4) A=1.1, (5) A=1, (6) A=0.9.

value, indicating that a breather is never generated. Between this critical value and a second one above which a breather is guaranteed to be generated, the increase of the noise intensity induces a rapid increase of the probability. In order to determine the appropriate amount of noise to generate a breather, we can define two critical values $\sigma_{10\%}$ and $\sigma_{90\%}$ which correspond to the noise intensity giving a probability of 10% and

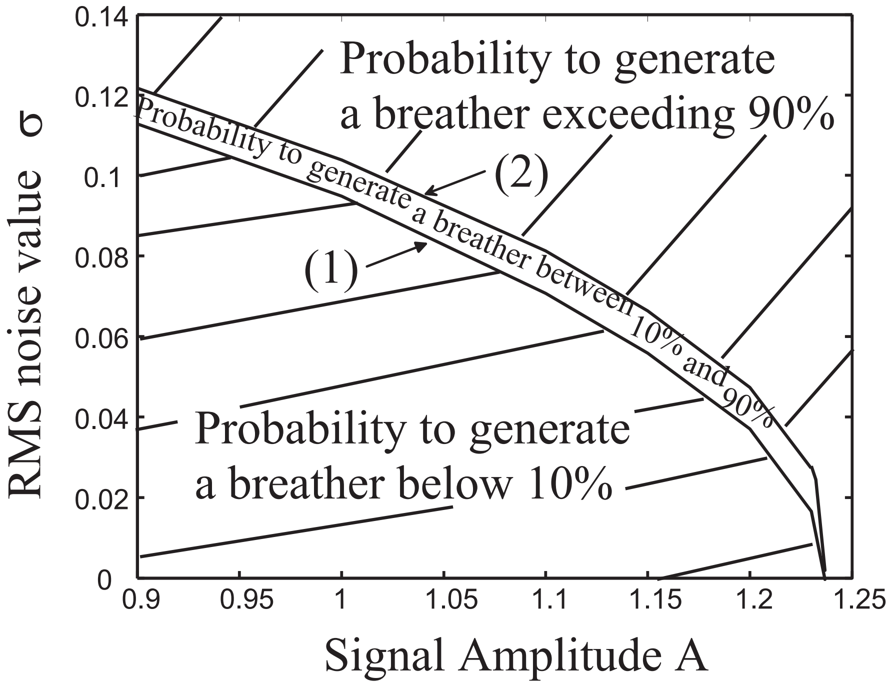


Figure 4. Bifurcation diagram of the sine-Gordon chain submitted to a noisy sinusoidal excitation. (1) Critical noise value $\sigma_{10\%}$ for which a breather is generated with 10% of probability versus the amplitude. (2) Critical noise value $\sigma_{90\%}$ for which a breather is generated with 90% of probability versus the amplitude. These two critical curves define three regions of parameters that allow to generate a breather with a given probability. Parameters: $N = 4000$, $m = 500$, $b = 140$, $\omega_0 = 1$, $c = 10$, $\Omega = 0.95$.

90%, respectively. These two quantities allow to plot the curves (1) and (2) of figure 4 and define three areas of parameters. Below the first curve, the amplitude of the driving and the noise intensity are too weak to generate a breather since the probability is below 10%. By contrast, beyond the second curve, the noisy sinusoidal driving is sufficiently strong to induce a breather in the medium with a probability exceeding 90%. Note that between the two curves of fig. 4, the noise intensity and the signal amplitude provided a breather with a probability in the range [10%; 90%].

4. Conclusion

In this paper, considering a discrete sine-Gordon chain, we have shown that the addition of a white noise to a sinusoidal excitation can induce the generation of breather modes in a parameter range where they do not appear without noise. Moreover, the probability to generate a breather has been numerically determined versus the noise intensity for different amplitudes of the driving. We have also provided a bifurcation diagram which reveals that noise can trigger the supratransmission effect below its deterministic threshold. Usually, the media that display the supratransmission effect also exhibit a bistable behavior resulting from nonlinearity and generating hysteresis properties

[25, 30, 31]. Since it has been proven that the bistable behaviour of nonlinear system can be used to realize stochastic detection [32], we trust that the sine-Gordon media which also shares a bistable feature could be used for signal processing purpose to enhance the detection of weak noisy signals. Therefore, this work could be the starting point for further investigations on nonlinear signal processing.

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